Australian Education Review

Teaching Mathematics: Using research-informed strategies

Peter Sullivan

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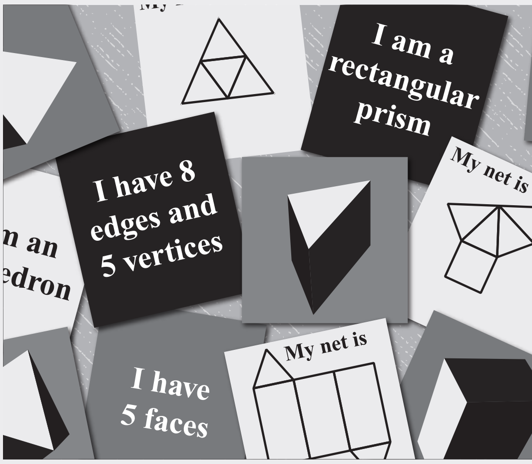
Foreword



I only recently took up a post at Monash University and so find myself between two worlds. On the one hand, I'm an 'old lag' in mathematics education, having been involved in research at King's College, London, for almost 20 years and on the other hand, a 'newbie' with respect to the culture and issues of mathematics teaching and research in Australia. Being asked to write this Foreword therefore comes at an apposite time – I'm still sufficiently 'alien' to bring what I hope is a fresh perspective to the research review, while at the same time it plunges me into thinking about the culture and these issues and themes, as they play out in Australia.

Sullivan frames his review by tackling head on the issues around the debate about who mathematics education should be for and consequently what should form the core of a curriculum. He argues that there are basically two views on mathematics curriculum – the 'functional' or practical approach that equips learners for what we might expect to be their needs as future citizens, and the 'specialist' view of the mathematics needed for those who may go on a study it later. As Sullivan eloquently argues, we need to move beyond debates of 'either/ or' with respect to these two perspectives, towards 'and', recognising the complementarity of both perspectives.

While coming down on the side of more attention being paid to the 'practical' aspects of mathematics in the compulsory years of schooling, Sullivan argues that this should not be at the cost of also introducing students to aspects of formal mathematical rigor. Getting this balance right would seem to be an ongoing challenge to teachers everywhere, especially in the light of rapid technological changes that show no signs of abating. With the increased use of spreadsheets and other technologies that expose more people to mathematical models, the distinction between the functional and the specialist becomes increasingly fuzzy, with specialist knowledge crossing over into the practical domain. Rather than trying to delineate the functional from the specialist, a chief aim of mathematics education in this age of uncertainty must be to go beyond motivating students to learn the mathematics that we think they are going to need (which is impossible to predict), to convincing them that they can learn mathematics, in the hope that they will continue to learn, to adapt to the mathematical challenges with which their future lives will present them.

Perhaps more challenging than this dismantling of the dichotomy of functional versus academic is Sullivan's finding that while it is possible to address both aspects current evidence points to neither approach being done particularly well in Australian schools. I would add that I do not think that is a problem unique to Australia: in the United Kingdom the pressure from National Tests has reduced much teaching to the purely instrumental.

Drawing on his own extensive research and the findings of the significant the National Research Council's review (Kilpatrick, Swafford & Findell, 2001), Sullivan examines the importance of five mathematical actions in linking the functional with the specialist. Two of these actions – procedural fluency and conceptual understanding – will be familiar to teachers, while the actions of strategic competence and adaptive reasoning, nicely illustrated by Sullivan in later sections of the review, are probably less familiar. The research shows that students can learn strategic competence and adaptive reasoning but the styles of teaching required to support such learning, even when we know what these look like, present still further challenges to current styles of mathematics teaching. These four strands of mathematical action – understanding, fluency, problem solving and reasoning – have been included the new national Australian mathematics curriculum.

The fifth strand that Sullivan discusses – productive disposition is, interestingly, not explicitly taken up in the ACARA model, for reasons not made clear in the review. If teachers have a duty to support learners in developing the disposition to continue to learn mathematics, then one wonders why this strand of action is absent. Of course, it may be that developing this is taken as a given across the whole of the curriculum. Looking back to the first version of England's national curriculum for mathematics in 1990 there was a whole learning profile given over to what might have been considered 'productive dispositions'. But difficulties in assessing learner progress on this strand led to its rapid demise in subsequent revisions of the curriculum. I hope that the Australian curriculum is not so driven by such assessment considerations.

In considering assessment, Sullivan points out that the PISA 2009 Australian data show that, despite central initiatives, the attainment gap between children from high and low SES home backgrounds seems to be widening. This resonates with a similar finding from the Leverhulme Numeracy Research Program (LNRP) in England that I was involved in with colleagues, data from which showed that the attainment gap had widened slightly, despite the claim that England's National Numeracy Strategy had been set up to narrow it (Brown, Askew, Hodgen, Rhodes & Wiliam, 2003). Improving the chances of children who do not come from supportive 'middle class' backgrounds seems to be one of mathematics education's intractable problems, particularly when addressed through large-scale, systemic, interventions. It is encouraging to read the evidence Sullivan locates as he explores the topic in Section 7 that carefully targeted intervention programs can make a difference in raising attainment for all.

The review contains interesting test items from Australia's national assessments, showing the range of student responses to different types of problem and how facilities drop as questions become less like those one might find in textbooks. As Sullivan points out, more attention needs to be paid to developing students' abilities to work adaptively – that is to be able to apply what they have previously learnt in answering non-routine questions – and that this in turn has implications for the curriculum and associated pedagogies.

Looking at definitions of numeracy, Sullivan makes the important argument that numeracy is not simply the arithmetical parts of the mathematics curriculum and is certainly not the drilling of procedural methods, as the term is sometimes interpreted. He points out that a full definition of numeracy requires greater emphasis be placed on estimation, problem solving and reasoning – elements that go toward helping learners be adaptive with their mathematics. Alongside this, Sullivan argues, an important aspect to consider in using mathematics is the 'social perspective' on numeracy: introducing students to problems where the 'authenticity' of the context has to take into consideration the relationships between people in order to shape solutions. For example, having students recognise that interpersonal aspects, such as 'fairness', can impact on acceptable solutions. A 'social perspective' is more than simply the application of previously learnt mathematics to 'realistic' contexts; it also generates the potential that using students' familiarity with the social context can lead to engagement with the mathematics. The researcher Terezhina Nunes makes a similar point when she talks about children's 'action schemas' – the practical solving of everyday problems – as providing a basis from which to develop mathematics (Nunes, Bryant & Watson, 2009). As she has pointed out, while young children may not be able to calculate with 3 divided by 4 in the abstract, few groups of four children would refuse three bars of chocolate on the basis of not being able to share them out fairly.

While Sullivan points to the importance of contexts needing to be chosen to be relevant to children's lives, I think we have to be cautious about assuming that any 'real world' context will be meaningful for all students. Drawing on 'everyday' examples that appeal to values and expectations that might be termed 'middle class' – such as mortgage rates, savings interests, and so forth - could prove alienating to some students, rather than encompassing or relevant.

At the time of writing, Finland is being reported in the press as having solved the 'problem' of difference, but commentators within Finland note that until recently the largely monocultural make-up of Finnish society meant that teachers' own backgrounds were very similar to those of the majority of students that they taught. As immigration into Finland has risen, with increased diversity within classrooms, so educators within Finland are far from confident that Finland will continue to maintain its high ranking in international studies as teachers work with students who come from backgrounds very different to their own. A key issue across the globe is how to broaden teachers' awareness of the concerns of families with whom they do not share similar backgrounds.

We need to remember that school mathematics has a 'social perspective' in and of itself and that some students will find meaning in contexts that are purely mathematical. Psychologist Ellen Langer (1997) refers to a 'mindful' approach to knowledge and has reminded us that human agency over choices is at the heart of most 'facts', including mathematical ones. For example take the classic representation of a quarter as one out of four squares shaded: engaging with this representation mindfully would mean being aware of the possibility that the image could equally well have been decided upon as the representation of one-third, by comparing the shaded part to the unshaded part. Indeed many students will 'read' such a diagram as one third. A social, or mindful, perspective reminds us that students who 'read' the diagram as ⅓ rather than ¼ are not simply 'misunderstanding' here, but are interpreting the diagram in a way that, in other circumstances, could be considered appropriate.

Nor should we dismiss the role of fantasy and imagination in young learners lives – a problem that is essentially a mathematical puzzle involving pigs and chickens may be just as 'meaningful' to some learners when the context is changed to aliens with differing numbers of legs, as it is in changing it to humans and dogs. Contexts can doubtless make more mathematics meaningful and more engaging to more learners, but no context will make all mathematics meaningful to all learners.

Sullivan further develops the issue of meaningfulness in his section on tasks, noting that students do have a diversity of preferences, and so affirming the importance of teachers providing variety in the tasks at the core of their mathematics lessons. I agree and would add that one of the great challenges in teacher preparation is helping teachers to recognise their interests (possibly, 'I definitely prefer the 'purely' mathematical over the 'applied' and the algebraic over the geometric') and to then step outside their own range of preferred problems, to broaden the range of what they are drawn to offer.

Part of developing a social perspective means looking at the opportunities for numeracy in other curriculum areas. All too often this is interpreted as numeracy travelling out into other curriculum areas, but Sullivan raises the important issue of making opportunities within the mathematics lesson to explore other aspects of the curriculum. Again, as Sullivan indicates, we should not underestimate the challenges that this places on all teachers, for whom adopting a collaborative approach to teaching may not be 'natural'. It is also not simply a case of identifying 'topics' that might lend themselves to a mathematical treatment, but of opening up conversations amongst teachers of different subjects about their views of the possible role of mathematics in their classes, together with how to introduce the mathematics so that there is consistency of approaches.

In Section 5 Sullivan clearly articulates the research and rationale underpinning six key principles that he argues underpin effective mathematics teaching. I want to comment on the trap of translating principles into practices in such a way that practical suggestions become so prescriptive that they are severed from the underlying principle being referenced.

One of Sullivan's principles is about the importance of sharing with students the goals of mathematics lessons. I'm old enough to have taught through a time when it was thought good practice to 'dress-up' mathematics so that, in my experience, children might not even have known that they were in a mathematics lesson. There is now no doubting that learning is improved when learners explicitly engage in thinking about what they are learning. In England, however, this quest for explicitness turned into a ritual of always writing the lesson objective on the board at the start of a lesson and students copying it down into their books. The LNRP data showed that while this may have been a positive framing for lessons, when routinely followed some unintended outcomes occurred. These included: focusing on learning outcomes that could most easily be communicated to students; lessons based on what seems obviously 'teachable'; the use of statements that communicated rather little in the way of learning outcomes, for example, 'today we are learning to solve problems' seems unlikely to raise much learner awareness. In many lessons observed as part of the LNRP evaluation, it would have been more valuable to have had a discussion at the end of the lesson to elaborate what had been learnt rather than trying to closely pre-specify learning outcomes at the beginning of a lesson.

In the final section of this research review, Sullivan summarises the implications for teacher education and professional development. As he indicates, there is still much work that needs to be done to improve mathematics teaching and learning. This research review makes a strong contribution to the beginning of that work.

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Contents

section

1

Introduction to the review

This review of research into aspects of mathematics teaching focuses on issues relevant to Australian mathematics teachers, to those who support them, and also to those who make policy decisions about mathematics teaching. It was motivated by and draws on the proceedings of the highly successful Australian Council for Educational Research Council (ACER) conference titled *Teaching Mathematics? Make it count: What research tells us about effective mathematics teaching and learning*, held in Melbourne in August 2010.

The review describes the goals of teaching mathematics and uses some data to infer how well these goals can and are being met. It outlines the contribution that numeracy-based perspectives can make to schooling, and describes the challenge of seeking equity of opportunity in mathematics teaching and learning. It argues for the importance of well-chosen mathematics tasks in supporting student learning, and presents some examples of particular types of tasks. It addresses a key issue facing Australian mathematics teachers, that of finding ways to address the needs of heterogeneous groups of students. It offers a synthesis of recommendations on the key characteristics of quality teaching and presents some recommendations about emphases which should be more actively sought in mathematics teacher education programs.

The emphasis throughout this Australian Education Review is on reviewing approaches to teaching mathematics and to providing information which should be considered by teachers in planning programs designed to address the needs of their students.

Underpinning perspective on learning

The fundamental assumption which informs the content of this review paper was also the dominant perspective on knowledge and learning at the *Teaching Mathematics? Make it count* conference, known as 'social constructivism'. In his review of social constructivism Paul Ernest described 'knowing' as an active process that is both:

> … *individual and personal, and that is based on previously constructed knowledge.*
>
> (Ernest, 1994, p. 2)

Basically, this means that what the teacher says and does is interpreted by the students in the context of their own experiences, and the message they hear and interpret may not be the same as the message that the teacher intended. Given this perspective, teaching cannot therefore be about the teacher filling the heads of the students with mathematical knowledge, but interacting with them while they engage with mathematics ideas for themselves.

An important purpose of the review paper is to review research on teaching mathematics currently being conducted in Australia, and to offer some suggestions about emphases in policy and practice. Some relevant international research and data are also reviewed, including papers presented by international researchers who presented at the ACER conference. Of course, effective teaching is connected to what is known about the learning of particular topics, but such research is not reviewed due to limitations of space.

It should be noted that little mathematics education research adheres to strict experimental designs, and there are good reasons for that. Not only are changes in learning or attitudes difficult to measure over the duration of most projects, but also the use of control groups among school children is prohibited by many university ethics committees. The projects and initiatives that are reported in this review paper include some that present only qualitative data or narrative descriptions, but all those chosen have rigorous designs and careful validity checks.

Structure of this review

Section 2 in this review paper summarises two perspectives on mathematics learning and proposes that the practical or numeracy perspective should be emphasised in the compulsory years, recognising that it is also important to introduce students to specialised ways of thinking mathematically. It describes the key mathematical actions that students should learn, noting that these actions are broader than what seems to be currently taught in mainstream mathematics teaching in Australia.

Section 3 uses data from national and international assessments to gain insights into the achievement of Australian students. It also summarises some of the issues about the decline in participation in advanced mathematics studies in Year 12, and reviews two early years mathematics assessments to illustrate how school-based assessments provide important insights into student learning.

Section 4 argues that since numeracy and practical mathematics should be the dominant focus in the compulsory years of schooling, teaching and assessment processes should reflect this. This discussion is included in the review since there is substantial debate about the nature of numeracy and its relationship to the mathematics curriculum. The basic argument is that not only are numeracy perspectives important for teaching and assessment in mathematics in the compulsory years, but also that they offer ways of thinking mathematically that are useful in other teaching subjects.

Section 5 lists six specific principles that can inform mathematics teacher improvement. It argues that these principles can be productively used and should be adopted as the basis of both structured and school-based teacher learning.

Section 6 describes and evaluates research that argues that the choice of classroom tasks is a key planning decision and teachers should be aware of the range of possible tasks, their purposes, and the appropriate pedagogies that match those tasks. This perspective should inform those who are developing resources to support mathematics teaching.

Section 7 argues that, rather than grouping students by their achievement, teachers should be encouraged to find ways to support the learning of all students by building a coherent classroom community and differentiating tasks to facilitate access to learning opportunities.

Section 8 addresses the critical issue, for Australian education, that particular students have reduced opportunities to learn mathematics. It summarises some approaches that have been taken to address the issue, including assessments and interventions that address serious deficiencies in student readiness to learning mathematics.

Section 9 proposes a framework that can guide the planning of teacher professional learning in mathematics, including four particular foci that are priorities at this time.

Section 10 is a conclusion to the review.

section 2

The goals of school mathematics

To define the goals of mathematics teaching, it is necessary to consider what mathematics is and does and what might be the purposes for teaching mathematics to school students. Drawing on key presentations at the *Teaching Mathematics? Make it count* conference, this section describes perspectives on the goals of mathematics teaching (which can be thought of as nouns) and contrasts these with what seems to be the dominant approach to teaching mathematics currently. Section 2 also describes key mathematical actions with which students can engage (which can be thought of as verbs). The basic argument is that the emphasis in school mathematics should be predominantly on practical and useable mathematics that can enrich not only students' employment prospects but also their ability to participate fully in modern life and democratic processes. This section also argues that students should be introduced to important mathematical ideas and ways of thinking, but explains that these mathematical ideas are quite different from the mathematics currently being taught even at senior levels of schooling. Finally, the section presents some data derived from international and national assessments on the mathematics achievement of Australian students, whose low achievement threatens their capacity to fully participate.

Two perspectives on the goals of mathematics teaching

There is a broad consensus among policy makers, curriculum planners, school administrations and business and industry leaders that mathematics is an important element of the school curriculum. Rubenstein (2009), for example, offers a compelling description of the importance of mathematics from the perspective of mathematicians, as well as the challenges Australia is facing due to the decline in mathematics enrolments in later year university mathematics studies. Indeed, the importance of mathematics is implicitly accepted by governments in the emphasis placed on monitoring school improvement in mathematics and in mandating the participation of Australian students in national assessment programs and reporting through the *MySchool* website (which can be accessed at http://www.myschool.edu.au/). Yet there is still an ongoing debate within the Australian community on which *aspects of mathematics* are important, and which aspects are most needed by school leavers.

On one side of the debate, commentators argue for the need to intertwine conventional discipline-based learning with practical perspectives, while those on the other side of the debate emphasise specifically mathematical issues in mathematical learning. And this debate is far from an 'academic' one, since to decide which path to follow will have an enormous impact on individual teachers and learners, and on what mathematical understandings are available to the broader society in subsequent years. This last point is discussed in greater detail in Section 3.

Part of the context in which this debate is being conducted is that schools are confronting the serious challenge of disengaged students. In their report on the national Middle Years Research and Development Project, Russell, Mackay and Jane (2003) made recommendations for reform associated with school leadership and systematic school improvement, especially emphasising the need for more interesting, functionally relevant classroom tasks which can enhance engagement in learning. This review paper argues that the last recommendation has particular resonance for mathematics teaching. Klein, Beishuizen and Treffers (1998) had previously described what forms such recommendations might take in the context of mathematics learning, and they connected the role such tasks had in better preparing school leavers for employment and for their everyday needs as citizens. Additionally, there is said by some to be a serious decline in the number of students completing later year university level mathematics studies, thereby threatening Australia's future international competitiveness and capacity for innovation. These claims feed calls for more mathematical rigour at secondary level, as preparation for more advanced learning in mathematics. Unfortunately, these claims are presented by the protagonists as though teachers must adopt one perspective or the other. This review argues that it is possible to address both functional relevance and mathematical rigour concurrently, but that neither perspective is being implemented well in Australian schools.

This debate between the functionally relevant perspective and that of mathematical rigour deals with both the nature of disciplinary knowledge and the nature of learning. And it is one which is being had in many countries. The debate presents as the 'Math War' in the United States of America (Becker & Jacobs, 2000). There are similar disputes in the Netherlands, and calls by various groups for mathematical rigour and the public criticism of their successful and internationally recognised *Realistic Mathematics Education* approach have been described by van den Heuvel-Panhuizen (2010).

Notwithstanding the strongly held views of those on both sides of this debate, both perspectives have a relevance to the content and pedagogies of mathematics programs in schools. Consequently, this review will maintain that curricula should encompass both, though with variation according to the learners' capacity. It will also argue that all students should experience not only practical uses of mathematics but also the more formal aspects that lay the foundation for later mathematics and related study. The key is to identify the relative emphases and the foci within each perspective, according to the learners.

In one of the major presentations at the *Teaching Mathematics? Make it count* conference, Ernest (2010) delineated both perspectives. He described the goals of the *practical* perspective as follows: students learn the mathematics adequate for general employment and functioning in society, drawing on the mathematics used by various professional and industry groups. He included in this perspective the types of calculations one does as part of everyday living including best buy comparisons, time management, budgeting, planning home maintenance projects, choosing routes to travel, interpreting data in the newspapers, and so on.

Ernest also described the *specialised* perspective as the mathematical understanding which forms the basis of university studies in science, technology and engineering. He argued that this includes an ability to pose and solve problems, appreciate the contribution of mathematics to culture, the nature of reasoning and intuitive appreciation of mathematical ideas such as:

> *… pattern, symmetry, structure, proof, paradox, recursion, randomness, chaos, and infinity.*

(Ernest, 2010, p. 24)

He argued for there being some aspects of mathematics that go beyond the practical everyday instrumental uses of mathematics for other purposes. These constitute an interesting domain of knowledge in its own right, and it is such aspects that Ernest categorised as specialised.

The terms 'practical' and 'specialised' are used throughout this review to characterise these two different perspectives. The importance of both perspectives is evident in the discussions which are informing the development of the new national mathematics curriculum. For example, *The Shape of the Australian Curriculum: Mathematics* (ACARA) (2010a) listed the aims of emphasising the practical aspects of the mathematics curriculum as being:

> *… to educate students to be active, thinking citizens, interpreting the world mathematically, and using mathematics to help form their predictions and decisions about personal and financial priorities.*

(ACARA, 2010a, p. 5)

The aims of the specialised aspects are described as being that:

> *… mathematics has its own value and beauty and it is intended that students will appreciate the elegance and power of mathematical thinking, [and] experience mathematics as enjoyable.*

(ACARA, 2010a, p. 5)

In other words, ACARA required the new national curriculum in mathematics to seek to incorporate both perspectives. The key issue rests in determining their relative emphases. In his conference paper, Ernest (2010) argued that, while it is important that students be introduced to aspects of specialised mathematical knowledge, the emphasis in the school curriculum for the compulsory years should be on practical mathematics. In their 2008 report, Ainley, Kos and Nicholas noted that, while fewer than 0.5 per cent of university graduates specialise in mathematics, and only around 40 per cent of graduates are professional users of mathematics, a *full 100 per cent* of school students need practical mathematics to prepare them for work as well as for personal and social decision making.

It is clear the appropriate priority in the compulsory years should be mathematics of the practical perspective. While the education of the future professional mathematicians is not to be ignored, the needs of most school students are much broader. The term 'numeracy' is commonly taken by Australian policy makers and school practitioners to incorporate the practical perspective of mathematical learning as the goal for schools and mathematical curricula. This review paper argues that an emphasis on numeracy should inform curriculum, pedagogy and assessment in mathematics and even in other disciplines, especially in the compulsory school years.

To consider the extent to which current common approaches to mathematics teaching incorporate these dual perspectives, one can do no better than review the *Third International Mathematics and Science Study* (TIMSS), which aimed to investigate and describe Year 8 mathematics and science teaching across seven countries. In the Australian component of this international study, 87 Australian teachers, each from a different school, volunteered and this cohort provided representative regional and sectoral coverage across all Australian states and territories. Each teacher in their mathematics class was filmed for one complete lesson. With respect to Australian teaching practices, Hollingsworth, Lokan and McCrae reported in 2003 that most exercises and problems used by teachers were low in procedural complexity, that most were repetitions of problems that had been previously completed, that little connection was made to examples of uses of mathematics in the real world, and that the emphasis was on students locating just the one correct answer.

> *Opportunities for students to appreciate connections between mathematical ideas and to understand the mathematics behind the problems they are working on are rare.*

(Hollingsworth, Lokan & McCrae, 2003, p. xxi)

Similarly, at the ACER conference, Stacey (2010) reported findings from a recent study in which she and a colleague interviewed over 20 leading educators, curriculum specialists and teachers on their perspectives on the nature of Australian mathematics teaching. She concluded that the consensus view is that Australian mathematics teaching is generally repetitious, lacking complexity and rarely involves reasoning.

Such mathematics teaching seems common in other countries as well. For example, Swan (2005), in summarising reports from education authorities in the United Kingdom, concluded that much mathematics teaching there consisted of low-level tasks that could be completed by mechanical reproduction of procedures, without deep thinking. Swan concluded that students of such teachers are mere receivers of information, having little opportunity to actively participate in lessons, are allowed little time to build their own understandings of concepts, and they experience little or no opportunity or encouragement to explain their reasoning. Ernest (2010) further confirmed the accuracy of these findings, even for university graduates, who feel that mathematics is inaccessible, related to ability rather than effort, abstract, and value free.

A necessary corollary to incorporating these dual perspectives in mathematics teaching and learning in pedagogy is a consideration of the ways that teachers might engage their students in more productive learning. The research strongly suggests that teachers incorporate both types of mathematical actions in tasks for their students to undertake when learning mathematics.

Five strands of desirable mathematical actions for students

In discussing the connections between the practical and specialised perspectives with classroom practice this review paper posits that both perspectives need to incorporate a sense of 'doing', that the focus should be on the mathematical actions being undertaken during the learning. To further delineate the scope and nature of the mathematical actions that students need to experience in their mathematical learning, and which apply equally to both the practical and specialised perspectives, the following text reviews some ways of describing those actions. Kilpatrick, Swafford and Findell (2001) established and described five strands of mathematical actions, and Watson and Sullivan (2008) then further refined these five strands as described in the following subsections.

Conceptual understanding

Kilpatrick et al. (2001) named their first strand 'conceptual understanding', and Watson and Sullivan (2008), in describing actions and tasks relevant for teacher learning, explained that conceptual understanding includes the comprehension of mathematical concepts, operations and relations. Decades ago, Skemp (1976) argued that it is not enough for students to understand *how* to perform various mathematical tasks (which he termed 'instrumental understanding'). For full conceptual understanding, Skemp argued, they must also appreciate *why* each of the ideas and relationships work the way that they do (which he termed 'relational understanding'). Skemp elaborated an important related idea based on the work of Piaget related to schema or mental structures. In this work Skemp's (1986) basic notion was that well-constructed knowledge is interconnected, so that when one part of a network of ideas is recalled for use at some future time, the other parts are also recalled. For example, when students can recognise and appreciate the meaning of the symbols, words and relationships associated with one particular concept, they can connect different representations of that concept to each other and use any of the forms of representation subsequently in building new ideas.

Procedural fluency

Kilpatrick et al. (2001) named their second strand as 'procedural fluency', while Watson and Sullivan (2008) preferred the term 'mathematical fluency'. They defined this as including skill in carrying out procedures flexibly, accurately, efficiently, and appropriately, and, in addition to these procedures, having factual knowledge and concepts that come to mind readily.

At the *Teaching Mathematics? Make it count* conference, Pegg (2010) presented a clear and cogent argument for the importance of developing fluency for all students. Pegg explained that initial processing of information happens in working memory, which is of limited capacity. He focused on the need for teachers to develop fluency in calculation in their students, as a way of reducing the load on working memory, so allowing more capacity for other mathematical actions. An example of the way this works is in mathematical language and definitions. If students do not know what is meant by terms such as 'parallel', 'right angle', 'index', 'remainder', 'average', then instruction using those terms will be confusing and ineffective since so much of students' working memory will be utilised trying to seek clues for the meaning of the relevant terminology. On the other hand, if students can readily recall key definitions and facts, these facts can facilitate problem solving and other actions.

Strategic competence

The third strand from Kilpatrick et al. (2001) is 'strategic competence'. Watson and Sullivan (2008) describe strategic competence as the ability to formulate, represent and solve mathematical problems. Ross Turner, in his presentation at the *Teaching Mathematics? Make it count* conference, termed this 'devising strategies', which he argued involves:

> *… a set of critical control processes that guide an individual to effectively recognise, formulate and solve problems. This skill is characterised as selecting or devising a plan or strategy to use mathematics to solve problems arising from a task or context, as well as guiding its implementation.*

(Turner, 2010, p. 59)

Problem solving has been a focus of research, curriculum and teaching for some time. Teachers are generally familiar with its meaning and resources that can be used to support students learning to solve problems. The nature of problems that are desirable for students to solve and processes for solving them will be further elaborated in Section 5 of this review paper.

Adaptive reasoning

The fourth strand from Kilpatrick et al. (2001) is 'adaptive reasoning'. Watson and Sullivan (2008) describe adaptive reasoning as the capacity for logical thought, reflection, explanation and justification. Kaye Stacey (2010) argued in her conference paper that such mathematical actions have been underemphasised in recent Australian jurisdictional curricula and that there is a need for resources and teacher learning to support the teaching of mathematical reasoning. In an analysis of Australian mathematics texts, Stacey reported that some mathematics texts did pay some attention to proofs and reasoning, but in a way which seemed:

> *… to be to derive a rule in preparation for using it in the exercises, rather than to give explanations that might be used as a thinking tool in subsequent problems.*

(Stacey, 2010, p. 20)

Productive disposition

The fifth strand from Kilpatrick et al. (2001) is 'productive disposition'. Watson and Sullivan (2008) describe productive disposition as a habitual inclination to see mathematics as sensible, useful and worthwhile, coupled with a belief in diligence and one's own efficacy. As the name of this strand suggests, this is less a student action than the other strands, but it remains one of the key issues for teaching mathematics, because positive disposition can be fostered by teachers, and possessing them does make a difference to learning. Its importance, especially with low-achieving students, will be further elaborated in Section 9 of this review.

Discussion of the five desirable actions

The first four of these actions are incorporated into *The Shape of the Australian Curriculum: Mathematics* and described as 'proficiencies' (ACARA, 2010a). The simplified terms of 'understanding', 'fluency', 'problem solving' and 'reasoning' are used in the document for ease of communication, but they encompass the range of mathematical actions as described above. Previously, the curricula of most Australian jurisdictions use the term 'working mathematically' to describe mathematical actions. ACARA (2010a) argued that the notion of 'working mathematically' creates the impression to teachers that the actions are separate from the content descriptions, whereas the intention is that the full range of mathematical actions apply to each aspect of the content. ACARA (2010a) describes these as proficiencies, and in addition to giving full definitions, also use these proficiency words in the content descriptions and the achievement standards that are specified for the students at each level.

All five of these sets of mathematical actions have implications for mathematics teaching of both the practical and specialised perspectives. As is argued in various places in this review paper, all five mathematical actions are important and contribute to a balanced curriculum. One of the challenges facing mathematics educators is to incorporate *each* of the mathematical actions described in this subsection into centrally determined and school-based assessments, to ensure that they are appropriately emphasised by teachers. This is made more difficult by the way in which fluency is disproportionately the focus of most externally set assessments, and therefore is emphasised by teachers especially in those years with external assessments, often to the detriment of the other mathematical actions.

Concluding comments

There are different and to some extent competing perspectives on the goals of teaching school mathematics, and there are differing ways of delineating the mathematics actions in which students can be encouraged to engage. This section has argued that the main emphasis in mathematics teaching and learning in the compulsory years should be on practical mathematics that can prepare students for work and living in a technological society, but that all students should experience some aspects of specialised mathematics. To experience such a curriculum would be quite different from the current emphasis on procedural knowledge that dominates much of the Australian teaching and assessment in mathematics.

Section 3 provides a further perspective on mathematics teaching in Australia through considering both national and international assessment data, and makes some comments on participation in post-compulsory mathematics studies.

section 3

Assessments of student mathematics learning

Section 2 discussed the dual foci of practical and specialised mathematics content, through the matrix of the five mathematical actions. As part of the consideration of the state of mathematics learning in Australia, and an appreciation of the degree to which Australian students are achieving the goals of mathematics spelt out in the previous section, Section 3 will first examine some findings about student achievement in those five mathematical actions from international assessments of student learning. It will then consider implications from changes in enrolments in senior secondary mathematics studies, and describe two important school-based interview assessment tools as strategies which may assist in achieving those goals.

Comparative performance of Australian students in international studies

Australia participates in a range of international assessment of mathematics achievement such as the *Programme for International Student Achievement* (PISA) which assesses 15-year-old students, and *Trends in International Mathematics and Science Study* (TIMSS), conducted in 2002 and 2007, which assessed students in Year 4 and Year 8.

Ainley, Kos and Nicholas (2008) analysed the results from the 2006 and 2009 PISA assessments. They reported that in the 2006 PISA study, only 8 out of 57 countries performed significantly better than Australia in mathematics. Australia's score was 520, behind countries like Finland (548) and the Netherlands (531). Even though not at the top of these international rankings, these results do not indicate the Australian schools, as a cohort, are failing. Indeed, the sample from the Australian Capital Territory scored 539, and the sample from Western Australia scored 531, which are close to the leading countries, though this also indicates that students in other jurisdictions are performing less well. Thomson, de Bortoli, Nicholas, Hillman and Buckley (2010), in commenting on the 2009 PISA mathematics results, noted that while the performance of Australian students had remained strong, the ranking of the full cohort of Australian students in mathematics had declined, and this decline was reported as being mainly due to a fall in the proportion of students achieving at the top levels. Thomson et al. (2010) reported that the proportion of students at the lowest levels was similar to previously, although the fraction of these from low socioeconomic groups had increased. The results of students in the Northern Territory and Tasmania were substantially lower than those from the other states and territory. This between-jurisdiction variation suggests that any initiatives to improve results overall will need to be targeted.

However, in the 2007 TIMSS study, certain groups of Australian students performed less well comparatively than those groups in some other countries. Sue Thomson (2010), in her paper at the *Teaching Mathematics? Make it count* conference, reported that at Year 4 level Australian students overall were outperformed by all Asian countries, and by those in England and the United States of America as well. Similarly at Year 8, Australian students were outperformed by countries with whom they had previously been level. An interesting aspect is the between-item variation in the comparative results. The Australian students performed better than the comparison countries on some items, worse on some, and much worse on others, especially those requiring algebraic thinking.

Overall, the international comparisons suggest that some Australian students have done fairly well, although there is a diversity of achievement between states, between identifiable groups of students (further discussed in Sections 7 and 8), and on particular topics. Manifestly, the trends are not encouraging. The implications of such results for policy, resource development and the structure of teacher learning opportunities are elaborated in Section 9 of this review paper.

Differences in achievement of particular groups of students

This subsection describes the extent of the differences in achievement among particular groups of students. The data which have been included in the Table 3.1 have been extracted from the 2009 report on the *Programme for International Student Assessment* (PISA) (Thomson et al., 2010), which compared the different achievement levels of Year 8 students, based on the socioeconomic background of their parents. The table compares the achievement of students whose parents were in the upper quartile of SES level with those in the lowest quartile. The achievement level data in the table were derived by using the highest reported level and combining the two lowest levels (that is levels 0 and 1).

Table 3.1: Percentages of Australian students by particular socioeconomic backgrounds and PISA mathematical literacy achievement

	At the highest level	Not achieving level 2
Low SES quartile	6	28
High SES quartile	29	5

(Data for this table compiled from PISA report, Thomson et al., 2010, p. 13)

The PISA report indicates that 'at the highest level' students can:

> *… conceptualise, generalise, and utilise information; are capable of advanced mathematical thinking and reasoning; have a mastery of symbolic and formal mathematical operations and relationships; formulate and precisely communicate their findings, interpretations and arguments.*

(Thomson et al., 2010, p. 8)

These students are ready to undertake the numeracy and specialised mathematics curriculum for their year level. In contrast, the students not achieving level 2 are not yet able to use basic procedures or interpret results. These students would experience substantial difficulty with the mathematics and numeracy curriculum relevant for their age and year level.

The data in Table 3.1 show that while there are some low SES students achieving at the highest level, and some high SES students achieving at the lowest levels, there is an obvious trend. Having high SES parents very much increases a student's chances of reaching the highest levels. Alarmingly, there is only one difference between the 2006 and 2009 data and it is that the percentage of low SES students in the lower achievement level has increased. The rest are identical. The percentage of students in this group increased from 22 per cent to 28 per cent.

In other words, despite government initiatives in the intervening years to create opportunities through education, the achievement of low SES background groups has worsened.

Emphasising the critical role that parental income has in predicting achievement as distinct from the type of school, Thomson et al. (2010) argued that when achievement is controlled for SES background, there is no difference between achievement of students from Independent, Catholic and Government schools, suggesting that government initiatives need to seek to reduce the disadvantage experienced by students according to their parental income level.

Thomson et al. (2010) identified other groups of students who have lower achievement levels than the comparison group. Students from remote schools have lower achievement than students from rural schools, who have lower scores than metropolitan students. The difference in mathematical literacy between students in remote areas from those in metropolitan localities represents one and a half years of schooling. Foreign born students have a similar profile of results to other students, regardless of SES and location, but first generation Australian children perform slightly better. This result challenges a number of conceptions about sources of inequality, in that in each of these three categories of student background there was a similarly wide diversity in achievement. It seems that it is not the languages background or the length of Australian residence that is important, but other factors, especially socioeconomic background of the parents. Indigenous students are, on average, 76 points below non-Indigenous students or the equivalent of two years of schooling. It should be noted there is significant within-cohort difference, as for many of these Indigenous students, location and SES are active factors, whereas for some, only one of these factors is present.

Thomson et al. (2010) also noted that boys outperformed girls. There is a substantial and cumulative research literature that has examined possible reasons for this discrepancy, which has varied over time (e.g., Forgasz & Leder, 2001; Forgasz, Leder & Thomas, 2003). In this assessment, boys achieved a 10-point advantage over girls in mathematics, which surprised and disappointed those authors, since previous PISA reports had indicated no significant gender-based differences in Australian mathematics achievement. Interestingly, the success of the funding of policy and initiatives implemented in the late 20th century to address the lower achievement of girls in mathematics (e.g., Barnes, 1998) are evidence that targeted interventions can improve the achievement of otherwise disadvantaged groups.

For all of these identified differences in achievement between particular groups, there is a need to develop strategies for overcoming these differences. Research and deep thought needs to be given to elaborate the ways curriculum and pedagogy may be contributing to, and should be used to counter the inhibiting factors. Some such strategies are elaborated in Sections 7 and 8.

Analysing student achievement on national assessments

A perspective on the progress of Australian students can be gained by an examination of student responses to items from national assessment surveys. Three items from the 2009 Year 9 Australia NAPLAN (National Assessment Program – Literacy and Numeracy) numeracy assessment (which disallowed calculators) will be analysed. The data for Victorian students, which demonstrate close to median achievement, can be taken as suitable for the purposes of this discussion. (The test papers are available at http://www.vcaa.vic.edu.au/prep10/naplan/testing/testpapers.html.) The first item on the Year 9 assessment to be considered here was presented as follows.

Figure 3.1

> Steven cuts his birthday cake into 8 equal slices. He eats 25% of the cake in whole slices. How many slices of cake are left?

(Victorian Curriculum and Assessment Authority, 2011a, p. 6)

For this question students had to provide their own number answer, and 85 per cent of Victorian students did so correctly. However, this indicates that 15 per cent of students provided no answer or a wrong one. This question requires mathematics that is included in the curriculum from many years prior to Year 9 and, even noting possible difficulties due to the formulation of the question, the lack of success of this group of students is cause for deep concern. If this is a realistic measure of the numeracy knowledge of Year 9 students, it also indicates the depth of challenge for school and classroom organisation and the pedagogical routines that are being used, since there are, in a notional class of 20,17 students who can do the task, but 3 students who cannot.

A second item for consideration follows.

Figure 3.2

> A copier prints 1200 leaflets. One-third of the leaflets are on yellow paper and the rest are on blue paper. There are smudges on 5% of the blue leaflets. How many blue leaflets have smudges?

(Victorian Curriculum and Assessment Authority, 2011a, p. 7)

The students could choose from four response options: 40, 60, 400, 800. Fifty-nine per cent of Victorian students selected the correct option. To respond requires students to calculate two-thirds of 1200, then calculate 5% of that, so the 59 per cent of students who responded correctly were performing at least moderately well. There are, though, 41 per cent, well over one-third, of students who could not choose the correct response from the four options and no doubt some who choose the correct response by guessing. Such students would experience substantial difficulty comprehending most of their Year 9 mathematics classes, and most certainly be unable to readily approach any subsequent mathematics studies or effectively use mathematics in their work and lives.

The third selected item was included in the assessment to measure readiness for specialised mathematics. The students were given an equation and asked a question.

Figure 3.3

> $2(2x - 3) + 2 + ? = 7x - 4$
>
> What term makes this equation true for all values of x?

(Victorian Curriculum and Assessment Authority, 2011a, p. 11)

Like the first selected item this one also required a 'write-in' answer and only 15 per cent of the Victorian students gave the correct response. This item requires students to use basic algebraic concepts (distributive law, grouping like terms). In other basic algebraic items on the same assessment, it appeared that between one-third and one-half of the students could respond correctly, suggesting that overall facility with basic algebra is low, even though algebraic ideas have been part of the intended curriculum for two years. The item in Figure 3.3 also involves a more sophisticated idea; that of comparing the equivalence of both sides of the equation, and, adding to the item difficulty, the format of this aspect of the item is unusual. The students' responses indicate that as a cohort, either their knowledge of such algebraic concepts, or their capacity to work with them *adaptively*, is low. The 85 per cent of students who could not adaptively respond to the unusual format would find algebraic exercises or problems requiring more than one step difficult, especially if the form of the problem is unfamiliar. It appears that most students are ill-prepared for later specialised mathematical studies requiring these concepts.

Interpreting mathematical achievement test results

Some care should be used in interpreting these results, as the student responses to these items may underestimate their capacity. This may be due to a range of factors: this was the last of the NAPLAN assessments that the students had to sit and, given the heated debate surrounding the assessment Year 9 students' motivation to perform at their best on such assessments, may have been questionable. Nevertheless, the response rates raise various concerns and provide some important insights that can be used to inform planning and teaching of mathematics.

The data indicate that some Year 9 students (around 3 students per class) are unable to answer very basic numeracy questions (Figure 3.1). It also seems that a substantial minority of Year 9 students (around 8 students per class) are not able to respond to practical mathematical items that require more than one step. The second selected item is representative of many realistic situations that adults will need to be able to solve, and so revision of the content, pedagogy, and assessment of mathematics and numeracy teaching may need examination to achieve the desired goals for teaching mathematics.

The percentage correct (15 per cent) achieved by Victorian students to the third algebra item indicates that only a small minority of students (3 students per class) can use algebra adaptively. This has implications for the way algebra is taught prior to Year 9 and subsequently. For example, inspection of mathematical texts indicates that the majority of introductory algebra exercises are introduced using the same presentation format, whereas it would be better for students to experience algebraic concepts in a variety of formats and forms of representation. This would enable students' knowledge to develop more as conceptual understanding, rather than as merely procedural fluency with problems, in a standardised format.

For students to demonstrate facility with items such as the three presented, they need to be flexible, adaptable, able to use the conceptual knowledge they have in different situations, to think for themselves, to reason, to solve problems, and to connect ideas together. In other words, teachers need to ensure that they provide opportunities for all students to experience all five of those mathematical actions described by Kilpatrick et al. (2001). Teachers need to emphasise such actions in their teaching and assess students' capacity for such actions progressively. The pedagogies associated with such teaching should be the focus of both prospective and practising mathematics teacher learning, along with knowledge of the curriculum and assessment. This is elaborated further in Section 9.

Participation in post-compulsory studies

Further insights into the mathematics achievement of Australian students can be gained by considering the participation rates in post-compulsory mathematics studies. These data not only give some measure of the success of the earlier teaching and learning experienced by students, but also indicate potential enrolment in mathematics studies at tertiary levels.

Levels of post-compulsory mathematical curricula offered

Despite the emphasis by some commentators on differences of provision across jurisdictions, substantial commonality in approaches to post-compulsory mathematics study was identified by Barrington (2006) who analysed the content of, and enrolments in, senior secondary mathematics courses across Australia and categorised three levels of subject choices by students.

The lowest level of mathematical study Barrington termed 'elementary'. The content of these subjects commonly includes topics such as business or financial mathematics, data analysis, and measurement, and in some places includes topics like navigation, matrices, networks and applied geometry, and most encourage the use of computer algebra system calculators. Specific examples of such subjects are General Mathematics (New South Wales), Further Mathematics (Victoria), and Mathematics A (Queensland). Each of these curricula choices count towards tertiary selection and are appropriate for participation in most non-specialised university mathematics courses, and for professional courses such as teacher and nurse education.

Barrington termed the next level of subjects 'intermediate'. Common subject names are Mathematical Methods, Mathematics, Mathematics B, Applicable Mathematics and Mathematics Studies. Common topics include graphs and relationships, calculus and statistics focusing on distributions. Some such subjects allow use of computer algebra system calculators in the teaching and learning, as do some offerings of the next level subject. These subjects are taken by students whose intention is to study mathematics at tertiary level, as part of courses such as Engineering, Economics and Architecture.

The top level of mathematics subjects is described as 'advanced' mathematics, with the most common descriptor being Specialist Mathematics. Other terms are Mathematics Extension (1 and 2), Mathematics C and Calculus. These subjects commonly include topics such as complex numbers, vectors with related trigonometry and kinematics, mechanics, and build on the calculus from the intermediate level subject. They provide the ideal preparation for those anticipating graduating in fields such as mathematics at tertiary level.

Changes in senior school mathematics enrolments

There is debate about the interpretation of the significance of the data regarding enrolments in post-compulsory mathematical courses.

Various reports have noted a decline in enrolments in the top two levels of senior school mathematics studies. There does seem to be a move by students over the last decade away from the higher level mathematics subjects. Both Forgasz (2005) and Barrington (2006) reported a decline in enrolments in the advanced and intermediate levels. Ainley et al. (2008) reported that over the period 2004 to 2007, after being more or less constant for the previous ten years, enrolments in Mathematics Extension in New South Wales declined from 22.5 per cent to 19 per cent, and in Victoria enrolments in Specialist Mathematics declined from 12.5 per cent to 9.8 per cent.

Substantial concern has been expressed in the community of university mathematicians at this enrolment decline. Rubenstein (2009) claimed that mathematics, as a community asset, is in a 'dire state' (p. 1). He noted that demand for mathematicians and statisticians is increasing (coincidentally thereby reducing the available number of those who might choose mathematics teaching as a career). He argued that Australia is performing poorly, with only 0.4 per cent of graduates having a mathematics major in their degree, compared to the OECD average of 1 per cent.

There is, however, another perspective on these data, which questions whether the changes in enrolments are a cause for concern. The proportion of final year secondary students who study at least one of these mathematics subjects is close to 80 per cent and has been constant over the period 1998 to 2008 (Ainley et al., 2008), due mainly to increases in the numbers of students taking the elementary level subjects. In other words, a significant majority of students completing Year 12 *are* studying a mathematics subject. Even though Barrington used the term 'elementary' for this level of subject, those involved in the design of the 'elementary' subject in Victoria, for example, argue that it is a substantial mathematics subject choice. Its rationale is described as being:

> *… to provide access to worthwhile and challenging mathematical learning in a way which takes into account the needs and aspirations of a wide range of students. It is also designed to promote students' awareness of the importance of mathematics in everyday life in a technological society, and confidence in making effective use of mathematical ideas, techniques and processes.*

(Victorian Curriculum and Assessment Authority, 2011b, p. 1)

An additional indication of the strength and suitability of this subject is that it is increasingly being set by university faculties as a prerequisite for professional courses at university. Further, despite the decline in enrolments in the intermediate and advanced level subjects over recent years, there were still around 23,000 students enrolled in advanced subjects in 2007 and 61,000 students enrolled in the intermediate option. These numbers ensure that there are sufficient potential applicants for the available places in tertiary studies, especially since hardly any courses, professional or otherwise, list the relevant advanced mathematics studies as a prerequisite for entry. Increasing the enrolments in the intermediate and advanced level subjects is hardly likely to redress the decline in those choosing to study mathematics at university. Therefore, this review paper takes the stance that, given current enrolments, there is limited need for concern about declining enrolments in mathematics at senior levels. Rather, the challenge is to encourage those students who do complete the intermediate and advanced level subjects to enrol in mathematics studies in their first year of university, and then continue those studies into later years.

School-based assessment of student learning

There have been substantial criticisms of the negative impact of externally prescribed assessments. Nisbet (2004) and Doig (2006) both analysed Australian teachers' use and responses to systemic assessments tests and concluded that teachers made inadequate use of data, and the assessments had a negative impact on classroom practice. Williams and Ryan (2000) made similar criticisms related to the use of such assessments by teachers in England. At the *Teaching Mathematics? Make it count* conference, Rosemary Callingham argued that assessment is more productive when seen as a teacher responsibility. She presented separate descriptions delineating assessment as 'for learning, as learning, and of learning'. She concluded:

> *Assessment is regarded as more than the task or method used to collect data about students. It includes the process of drawing inferences from the data collected and acting on those judgements in effective ways.*

(Callingham, 2010, p. 39)

Similar comments were made by Daro (2010) in his conference presentation. Each of these scholars has argued for more school-based assessments of student learning for diagnostic purposes. While there are many ways for teachers to assess their students' learning in mathematics, it is useful to examine some interview assessments that have been found to produce helpful information for teachers. There are two well-designed school assessments that have been in use for some time and continue to be used widely by schools even after explicit funding has been removed.

The first diagnostic assessment is an individually administered structured interview, implemented by classroom teachers and supported by classroom resources, *Count Me In Too*. It was published by the NSW Department of Education and Training (2007) after many years of development and drawing on strong theoretical principles and detailed evaluations (Wright, Martland, & Stafford, 2000). It focuses on strategies that children use in solving arithmetic tasks. Both in professional learning sessions associated with its implementation and in its supporting manuals, classroom teachers are offered structured support to interpret the responses of the students and to devise ways of addressing deficiencies in the readiness of those students.

A similar interview assessment, developed as part of the Victorian *Early Numeracy Research Project* (Clarke, Cheeseman, Gervasoni, Gronn, Horne, McDonough, Montgomery, Roche, Sullivan, Clarke, & Rowley, 2002), was designed as a research tool to collect student mathematical assessment data over the first three years of schooling. To address the diversity of needs on school entry, a particular set of questions was developed, initially for use by the researchers and subsequently included as support for the teachers when using the interview assessment data. The evidence from the Victorian interview is that the early assessment of students is sufficiently powerful that schools and teachers are willing to overcome the organisational challenges of conducting one-on-one interviews. Clarke et al. reported that the:

> *… interview enabled a very clear picture of the mathematical knowledge and understandings that young children bring to school, and the development of these aspects during the first year of school. Most Prep children arrive with considerable skills and understandings in areas that have been traditional content for this grade level. As acknowledged by many trial school teachers, this means that expectations could be raised considerably in terms of what can be achieved in the first year.*

(Clarke et al., 2002, p. 25)

Such interview assessments have potential for informing teachers about the teaching and learning of numeracy and mathematics. The more a teacher knows about the strengths of a student, the better the teacher can facilitate the student's learning. Appropriately constructed school entry assessments, along with adequate school and system support for teachers to administer the assessments, and associated teacher professional development, can assist teachers in supporting the learning of students. Even though the focus in the Clarke et al. research had been on identifying students who may start school behind, which is of course critical in that they are likely to get further and further behind if their needs are not identified, these initial assessments also identify those students who are above the expected levels on entry, allowing teachers to extend their learning in positive ways.

Concluding comments

In this section student achievement results from international and national assessments were presented. The findings indicated that while some Australian students are doing well, others seem unprepared for the demands of mathematics study in the later secondary years. The findings have important implications for the pedagogies used in schools and for teacher learning initiatives. Some illustrative items from NAPLAN assessment were presented that illustrate how data can be used to inform decisions on curriculum and pedagogy for particular students.

The argument was presented that even though there has been a decline in enrolments in intermediate and advanced subjects at senior secondary level, there are still substantial numbers completing such studies. To better understand the reasons behind the decline in participation in university mathematics studies, further investigation is required.

Two particular school-based assessment instruments were described to emphasise that much assessment should be school-based and directed toward improvement rather than system monitoring.

Section 4 elaborates further ways that a numeracy/practical mathematics perspective can and should inform curriculum and teaching in Australian schools.

Numeracy, practical mathematics and mathematical literacy

section 4

Section 2 of this review paper argued that *practical* mathematics should be the major focus of mathematics teaching in the compulsory school years. The term 'numeracy' is most commonly used in Australia to encapsulate this practical perspective, while the term 'mathematical literacy' is used in the same way in many other countries and in international assessments such as PISA. Section 3 draws on common definitions of numeracy, in part to clarify the way the term 'numeracy' is used in this review paper, and also to elaborate three arguments. They are:

- that numeracy has particular meanings in the context of work, and these meanings have implications for school mathematics curriculum and pedagogy
- that there is a numeracy dimension in many social situations that can productively be addressed by mathematics teachers
- that numeracy perspectives can enrich the study of other curriculum subjects.

This discussion is included in the review paper since there is substantial debate about the meaning of the term 'numeracy' and ways that numeracy perspectives can contribute to curricula and teaching generally, and in mathematics.

Defining numeracy

The term 'numeracy' is used in various contexts and with different meanings, such as the following:

- as a descriptive label for systemic mathematical assessments
- in subsequent reporting to schools and parents
- as the name of a remedial subject
- to describe certain emphases in the mathematics curriculum and in other disciplines.

There is a diversity of opinions expressed on the nature of numeracy, ranging from those of some mathematicians who claim that numeracy does not exist, to some educators who claim it is synonymous with mathematics; and others who argue that the term 'numeracy' refers just to the *use* of mathematics in practical contexts.

The Australian Government Human Capital Working Group, concerned about the readiness for work of some school leavers, commissioned the National Numeracy Review (NNR). The review panel, which included leading mathematics educators, initially used the following definition of numeracy:

> *Numeracy is the effective use of mathematics to meet the general demands of life at school and at home, in paid work, and for participation in community and civic life.*

(Ministerial Council on Education, Employment, Training and Youth Affairs, 1999, p. 4)

The NNR report extended that definition to argue that:

> *… numeracy involves considerably more than the acquisition of mathematical routines and algorithms.*

(National Numeracy Review, 2008, p. xi)

However the NNR report, with its imprecise though well-intentioned definition, had little impact on school curricula, but the matter remained one of great importance to practitioners. This review paper prefers the more helpful clarification, which had been developed by *Australian Association of Mathematics Teachers* (AAMT, 1998) after extensive consultation with its members and a special purpose conference. This clarification contended that numeracy is:

> *… a fundamental component of learning, discourse and critique across all areas of the curriculum.*

(Australian Association of Mathematics Teachers, 1998, p. 1)

While the NNR definition sees numeracy as a subset of mathematics, the AAMT clarification argues that it is more. The AAMT affirmed that numeracy involves a disposition and willingness:

> *… to use, in context, a combination of: underpinning mathematical concepts and skills from across the discipline (numerical, spatial, graphical, statistical and algebraic); mathematical thinking and strategies; general thinking skills; [and] grounded appreciation of context.*

(Australian Association of Mathematics Teachers, 1998, p. 1)

This clarification and definition, when taken in full, conceptualises numeracy as informing mathematics, but it also goes beyond mathematics, having direct implications in life and in other aspects of the curriculum. In Section 3, the term 'numeracy' was used to encapsulate and include all of the elements of *practical* mathematics, but it made the distinction that numeracy is different from the learning of the *specialised* mathematics that forms part of the goals of schooling. Throughout this review paper the view will be put that the meaning of numeracy goes beyond specialised mathematics.

The contention in this review paper is that, while there are some situations that require only practical mathematics for solution, and some aspects of mathematics that have limited or no practical use although they are still valuable and important to the field and to learners, most real-life situations have some elements of both practical and specialised mathematics. This contention is exemplified in the many commentaries on the contribution of practical realistic examples to the learning of specialised mathematics (e.g., Lovitt & Clarke, 1988; Peled, 2008; Perso, 2006; Wiest, 2001). Most of these commentaries contain suggestions ranging from teachers using such examples to illustrate the relevance of mathematics to students' lives to recommendations that teachers use realistic contexts that illustrate the power of mathematics.

Therefore in the following discussions, the term 'numeracy' is used in the same way as it is in commonly used in curriculum and assessment policy, and it includes the meaning attributed to the term 'practical mathematics' in the first subsection of Section 3. This section will seek to still further extend the notion of numeracy and the following subsections elaborate on this.

Work readiness and implications for a numeracy curriculum

While not the only purpose of schooling, teachers at least need to consider preparing young people for the demands of employment and the exigencies of adult life. Consideration of the numeracy demands of work-readiness can inform not only the content of school curricula, but also pedagogical approaches used.

Over the past two decades there have been many studies of out-of-school numeracy practices of adults. Some have sought evidence of the use (or not) of recognised school mathematics topics in the workplace and society (FitzSimons, 2002). Others have examined the thinking processes used in particular contexts, known as 'situated cognition'. Lave (1988), for example, observed various groups of people at work and showed that the mathematical knowledge and skills utilised, for example by shoppers and weight watchers, bore little resemblance to the mathematical routines, procedures and even formulae taught in school. This research indicates that the relevance, location and teaching of many topics in school mathematics curricula need to be reconsidered, especially in the context of the argument for prioritising practical mathematics made in Section 2.

In recent years, several large-scale studies of numeracy in the workplace, in the United Kingdom (Bakker, Hoyles, Kent, & Noss, 2006), and in Australia (Kanes, 2002; FitzSimons & Wedege, 2007), have confirmed Lave's findings. Additionally, Zevenbergen and Zevenbergen (2009) have drawn attention to ways that young people use numeracy in their school work. Zevenbergen and Zevenbergen found that young workers did not use formal school mathematics even when solving problems involving measuring or proportion and ratios, but, instead, relied on the use of intuitive methods, only some of which were workplace specific. While Zevenbergen and Zevenbergen were critical of emphases in curricula on mathematics content that is irrelevant in workplaces, they also argued that such consideration of work demands has implications for the ways that mathematics is taught. They proposed that a greater emphasis on estimation, problem solving and reasoning, and a lesser emphasis on the development of procedural skills would assist in an increase in the relevance of mathematics learning to the workplace.

Collectively, these findings have important implications for the numeracy needs of future Australian citizens and contribute to an understanding of what needs to be emphasised in mathematics curricula and learnt by students for their work-readiness. The research indicates that since informed judgements about money, safety and accuracy are required in workplaces, workers need knowledge that is flexible and adaptable. The research also indicates that the context in which the mathematics is used is critical, that students need to be able to apply different disciplines simultaneously, that communication is important and that students should learn to use non-standard methods as well as the standard mathematical processes.

Interestingly, approaches that incorporate mathematics within practical contexts may well have the effect of engaging more students in learning numeracy and mathematics. Sullivan and Jorgensen (2009), for example, reported various case studies in which students saw tasks that were presented as part of a contextualised approach as relevant and accessible and were willing to invest the effort involved in learning the relevant numeracy. It is noted that many of these findings emphasise the need for the breadth of the mathematical actions recommended by Kilpatrick et al. (2001) that were described in Section 2.

A social perspective on numeracy

This subsection suggests that there are aspects of social decision making that can extend the ways that numeracy perspectives can enrich the school curriculum.

Teachers who pose to their students tasks which are placed within a clear social realistic context enable students to exercise some real-life experience as they consider and solve the tasks. Such an approach has the dual advantages of, on one hand, preparing students for life challenges and, on the other hand, illustrating that numeracy and mathematics can be useful for them in their lives.

Consider the following sample problems, suitable for upper primary students, the first two of which are adapted from Peled (2008).

Figure 4.1

> Julia and Tony decided to buy a lottery ticket for $5.
>
> Tony only had $1 on him so Julia paid $4.
>
> **Question 1:** If they got $20 back as a prize, what are some possible options for how they should share the prize?
>
> **Question 2:** If they won $50,000, what are some possible options for how they should share the prize?

A solution to Question 1 which focuses solely on the mathematical concept of proportionality would suggest that Julia should get $16. But another view could be argued; that since they are friends they should share the prize equally, perhaps after returning the original investment. The point is that there are mathematical and non-mathematical factors operating in the solutions to such questions, and the relative weight given to such factors makes the pathways to some solutions, as in many social situations, less than certain or clear-cut. In Question 2 the factors are the same, but the dollar scale of the outcome grows if the non-mathematical factors take precedence in solving the problem.

It is relevant for teachers to allow students to explore such examples from both mathematical and social perspectives. At least part of the function of the consideration of such tasks is in developing in students the orientation and capacity for explaining their reasoning. There are many situations in life in which disputes arise involving measurements (mainly money) and finding resolutions to such disputes is a key life skill, a key aspect of which is justifying one's reasoning.

The following example, also adapted from Peled (2008), raises similar issues.

Figure 4.2

> Julia and Penny went shopping for shoes. Julia selected two pairs, one marked at $120, and the other at $80. Penny chose a pair for $100. The shop offers a discount where shoppers get three pairs for the price of two.
>
> **Question:** What are some possible options for how much Julia and Penny should each pay?

Again responses can reference both mathematical and social elements and, depending on which elements are selected, quite different outcomes will result. When tackled from a social perspective the problem requires thinking about aspects of the mathematical aspects of the ratios involved. These various types of ratios are not trivial and can be used subsequently as examples of formal approaches to solving proportionality tasks.

This social approach can be an ideal way to engage students in interpreting things mathematically, especially if they are not naturally orientated to do so. Consider the following sequence of problems.

Figure 4.3

> Three people went on a holiday, a couple and a single person. They rented a two-bedroom apartment for $600 per week.
>
> **Question:** What are some possible options for how much each person should pay?

When viewed from a social perspective, one possibility is that there are three people so they should pay $200 each. It is also feasible to argue there are two bedrooms so the couple should pay $300 and the single person $300. As it stands, the question has both social and numeracy dimensions, and it also has the potential to open the door to some generalised mathematical thinking. For example, the following problem could be posed:

Figure 4.4

> Three couples and four single people rented a seven-bedroom lodge for $1400.
>
> **Question:** What are some possible options for how much each person should pay?

This version has the effect of extending the initial problem beyond the obvious and now there is a need for a more mathematical consideration of the options. This could be extended further by considering the following problem:

Figure 4.5

> x couples and y single people rented a lodge with plenty of bedrooms for $\$z$.
>
> **Question:** What are some possible options for how much each person should pay?

Such a sequence, using the same context, involves progressively increasing the complexity of problems as was proposed by Brousseau and Brousseau (1981), and shows how numeracy examples can lead to consideration of mathematical ways of representing situations. Indeed this social numeracy approach can provide an entry to mathematical thinking rather than being an application of it.

Each of these problems requires consideration of aspects beyond an arithmetical interpretation of the situation. The problems can be adapted so they are relevant to students, illustrate an explicit social dimension of numeracy, emphasise that some numeracy-informed decisions are made on social criteria, and that in many situations there can be a need to explain and even justify a particular solution. Such problems can also provide insights into the way that mathematics is used to generalise such situations.

Jablonka (2003), in an overview of the relationship between mathematical literacy and mathematics, argued for mathematics teachers to include a social dimension in their teaching. She suggested that numeracy perspectives can be useful in exploring cultural identity issues, and the way that particular peoples have used numeracy historically, as well as critical perspectives that are important not only for evaluating information presented in the media (an example of this is the arguments presented on each side of the global warming debates), but also for arguing particular social perspectives (for example, the extent to which Australia could manage refugees seeking resettlement). Numeracy perspectives could shed light on a broad range of public issues ranging from personal weight management, to health care (such as evidence used for and against public immunisation policies), to investments in stock and shares, to comparing phone plans, and so on.

Such contexts can be chosen to maximise relevance to students' lives, therefore making their learning of mathematics more meaningful for them and hopefully increasing their engagement with mathematical ideas. Such problems can even illustrate connections to other domains of knowledge, as is elaborated in the next subsection.

Numeracy in other curriculum areas

Another way that numeracy perspectives can enrich learning is in their incorporation into other aspects of the curriculum. This subsection will provide examples of how this might be achieved. In the case of primary school teachers, who in Australian schools teach all subjects to their class, this is mainly a matter of them being aware of potential links and finding ways of building connections across different domains of knowledge. However, for secondary teachers, who are subject specialists, incorporating numeracy perspectives into subjects other than mathematics is something of a challenge for two reasons. First, teachers of other curriculum areas are sometimes not convinced that quantitative perspectives illuminate the issues on which they focus. Second, many teachers who are specialists in non-mathematics subjects are neither confident nor skilled in approaches to working with students to model or explain the relevant numeracy. For the former, this is a matter of raising awareness. For the latter, some processes for specifically supporting such teachers will be necessary.

The following examples indicate how teachers of other curriculum areas might benefit from incorporating numeracy perspectives. The examples, which draw heavily on ACARA (2010a), predominantly apply to secondary schools by virtue of the topics, but the pedagogical approaches implied are also relevant for primary school teachers.

One topic with serious social consequences that is routinely discussed in the media is that of population planning for Australia, which includes the related issues of immigration. In geography, where for example this 'topic' may be addressed, a capacity to appreciate the relevant numeracy is critical to being able to interpret population flows and the impact of immigrants, including refugees, on population changes. To consider these issues, students will need to have data on the size of the Australian population, compared with data on net immigration inflow, the fraction of that net inflow that is the result of applicants for asylum, and perhaps comparison of those figures with other similar countries. All of these require collection and consideration of the relevant data and a capacity to manipulate the figures appropriately. While the basic skills required for making such calculations or estimates will be an outcome of effective mathematics teaching, consideration of the issues is clearly within the curricular remit of the geography teacher. The geography and mathematics teachers can both benefit from collaboration on such issues. The geography teacher can learn how to better present the data which illustrates the relevant ratio comparisons, and the mathematics teacher can benefit, through listening to their colleagues' thinking and description of their ways of dealing with data from within the discipline of geography.

In English literature study, the meaning and exegetical analysis of texts can be enriched by being more precise about the numeracy dimensions mentioned in the writing. For example, to truly understanding the scale of fortune that Jane Austen says that a man should amass before proposing to a woman, some comparative wealth figures from different levels of society 200 years ago, and comparative income rates from then to the present, converted to current Australian dollar values, would enhance students' appreciation of Austen's assertion.

There are, of course, approaches to the teaching of literacy that could profitably be incorporated into numeracy teaching, such as the teacher and students reading the text together, highlighting words that are important for mathematical meaning, writing key words on the board and saying them together, suggesting synonyms for difficult words, and so on. Again there are many instances of such possibilities, of collegial cooperation, from which both literacy and mathematics teachers would benefit.

In the subject history, students consider elapsed time, not only over large time periods such as, for example, the comparative length of Indigenous and immigrant settlement, but also over shorter periods, such as the chronological sequence of 20th century events. Mathematical tools and models are useful for explicating these periods of time. Both history and mathematics teachers can benefit from collaboration. History teachers are best placed to comment on the significance of such comparisons, and mathematics teachers are able to inform the calculations and even suggest appropriate models that can be used. Other topics for which a numeracy perspective would enhance the learning of history is in appreciation of large numbers, such as in population comparisons, trends in population over times, and experience of visualisation of space and places.

In science, students in the middle and senior secondary years perform calculations related to concentrations, titrations and unit conversions. Practical work and problem solving across all the sciences require the use of a range of measurements, capacity to organise and represent data in a range of forms and to plot, interpret and extrapolate through graphs. This also requires students to estimate, solve ratio problems, use formulae flexibly in a range of situations, perform unit conversions, use and interpret rates, scientific notation and significant figures. These concepts are better taught by the science teacher in the context of the science being learned, but without the appropriate pedagogies, the numeracy opportunities might be restricted to the learning of simply techniques. As with the other curriculum areas, there is clearly both a need for, and opportunities in, collaboration between mathematics teachers and those in other subjects to enrich the study of the context and the numeracy that can enrich study of other disciplines. Such cross-curricular approaches model to students ways that numeracy skills will be useful to them in many aspects of their future work and private lives.

Of course learning in many subjects is enhanced through the effective use of statistics. These should, of course, build on the concepts developed in mathematics classes, but the use of statistics in other contexts also needs to be considered by the physical education, domestic science, or technology teachers, for example. This requires collaboration and goodwill between the mathematics teachers and the teachers of those other subjects.

Concluding comments

This section has argued that numeracy is much more than a subset of mathematics. It also offers an important focus for school mathematics at all levels, in terms of preparation for the workplace, and also in connecting learners with the relationship between some social decisions and a mathematical analysis of the possibilities. A third focus of numeracy learning is to enrich the study of other curriculum areas.

In terms of the Australian curriculum, numeracy can offer examples and problems that connect the students with the mathematics they need to learn. It also provides explicit rationales and encouragement for primary teachers to incorporate/integrate mathematical learning across a wide range of subject areas and for secondary teachers to communicate with colleagues across subject boundaries. Basically, numeracy perspectives encourage students to see the world in quantitative terms, to appreciate the value and purpose of effectively communicating quantitative information, and to interpret everyday information represented mathematically. Adopting numeracy approaches in mathematics teaching can enable students to better anticipate the demands of work and life, and this has implications for curriculum, pedagogy and assessment. Incorporating numeracy perspectives in the teaching and learning of other disciplines can enrich students' understanding of those disciplines.

section 5

Six key principles for effective teaching of mathematics

This section follows on from the discussion of the goals of teaching mathematics and the data available on the mathematical achievement of Australian students. Having established the personal and social value of having mathematical understanding and some clarity about the skills current in the cohort of Australian mathematics students, the discussion now moves to what schools and teachers need to know and be able to do in order to address the shortfall between the required/desired and the demonstrated learning outcomes.

This section draws on research findings and other sets of recommendations for teaching actions, to present a set of six principles that can guide teaching practice. As the title of the *Teaching Mathematics? Make it count* conference indicates, there is the conviction that teaching mathematics well, in such a way as to make it count, is a worthwhile and reasonable proposition. This section presents a set of six principles of teaching mathematics which are specific to mathematics, but which are also based on sound general pedagogic principles that can relate to all curriculum areas. These principles are re-enforced by much of the research and the advice that follows in this paper. Overall, the review paper posits that they should be the focus for teacher education and professional learning in mathematics, which is addressed in Section 9.

The development of this review paper's six principles was partly motivated by the various lists of recommended practices from Australian education systems such as *Productive Pedagogies* (Department of Education and Training, Queensland, 2010) and *Principles of Learning and Teaching* (Department of Education and Early Childhood Development, Victoria, 2011) which are intended to inform teaching generally. Such lists are long and complex, and this author suspects that mathematics teachers experience difficulty in extracting the key recommendations for their particular practice. For example, one such set of recommendations is the South Australian *Teaching for Effective Learning Framework* (Department of Education and Child Services, South Australia, 2010), which lists four domains and 18 sub-domains. Some of the sub-domains are helpful, such as: build on learners' understandings; connect learning to students' lives and aspirations; communicate learning in multiple modes; support and challenge students to achieve high standards; and build a community of learners. There are others that are far from clear, such as: explore the construction of knowledge; negotiate learning; and, teach students how to learn. It is suspected that such recommendations provide general rather than specific support for mathematics teachers, and do not seem likely to prompt or motivate improvement in mathematics teaching practices.

While informed by such frameworks, the six principles for teaching mathematics defined and described in this review paper draw on particular national and international research reviews and summaries of recommendations about mathematics teaching. For example, this review paper's six principles for teaching mathematics incorporate key ideas from an early set of recommendations for mathematics teaching published by Good, Grouws and Ebmeier (1983), who synthesised results related to the effective teaching literature of the time.

The set of six principles for teaching mathematics also draws on Hattie (2009) who analysed a large number of studies that provide evidence about correlates of student achievement. He reported the effect size of a wide range of variables related to teachers, class grouping, and teaching practices, noting that identifying higher effect sizes is important since almost any intervention results in some improvement.

The six principles are also based on recommendations from Swan (2005) who presented a range of important suggestions, derived from earlier studies of teacher learning and classroom practice, on how teaching could move from promoting passive to active learning, and from transmissive to connected and challenging teaching.

Clarke and Clarke (2004) developed a similar set of recommendations, arising from detailed case studies of teachers who had been identified as particularly effective in the Australian *Early Numeracy Research Project*. Their list is grouped under ten headings and 25 specific actions. While their list was drawn from research with early years mathematics teachers, the headings and actions listed are applicable at all levels.

Similarly, this review paper's six principles also draw on Anthony and Walshaw's (2009) detailed best evidence synthesis which reviewed important research on mathematics teaching and learning, from which they produced a list of ten pedagogies, which they argued are important for mathematics teaching.

The following text presents the six principles, along with some indication of the impetus for each principle, written in the form of advice to teachers.

Principle 1: Articulating goals

This principle is elaborated for teachers as follows:

> Identify key ideas that underpin the concepts you are seeking to teach, communicate to students that these are the goals of the teaching, and explain to them how you hope they will learn.

This principle emphasises the importance of the teacher having clear and explicit goals that are connected to the pedagogical approach chosen to assist students in learning the goals. One of Hattie's chief recommendations (2009), which had earlier been elaborated in Hattie and Timperley (2007), was that feedback is one of the main influences on student achievement. The key elements of feedback are for students to receive information on 'where am I going?', 'how am I going?', and 'where am I going to next?' To advise students of the goals and to make decisions on pathways to achieving the goals interactively, requires teachers to be very clear about their goals. This is what Swan (2005) described as 'making the purposes of activities clear' (p. 6), and what Clarke and Clarke (2004) proposed as 'focus on important mathematical ideas and make the mathematical focus clear to the children' (p. 68).

This principle also reflects one of the key goals in *The Shape of the Australian Curriculum: Mathematics* (ACARA, 2010a), which argued for the centrality of teacher decision making, and for the curriculum to be written succinctly and specificly. This is precisely so that teachers can make active judgements on the emphases in their teaching. The flexibility in the modes of presentation of the content descriptions also indicates to teachers that their first step in planning their teaching is to make active decisions about their focus, and to communicate that focus to the students.

In particular, according to the thinking underpinning Principle 1, it is assumed that teachers would specifically articulate the key ideas/concepts to be addressed in the lesson before students begin, even writing the goals on the board. It is also expected that the students will learn, through working on a task, listening to the explanations of others, or by practising mathematical techniques.

Principle 2: Making connections

This principle is elaborated for teachers as follows:

> Build on what students know, mathematically and experientially, including creating and connecting students with stories that both contextualise and establish a rationale for the learning.

Relevant issues addressed earlier in this review have included the importance of practical mathematics and a presentation of a broader perspective on numeracy. Examples of tasks that emphasised relevance are analysed in the next section and the critical importance of connecting learning with the experience of low-achieving students in Section 8.

John Smith (1996), in a synthesis of recommendations for teachers, argued that using engaging tasks can assist teachers in achieving all of these goals. Here is a maths problem which is commonly posed as:

Figure 5.1

> A farmyard has pigs and chickens. There are 10 heads and 26 legs.
>
> **Question:** How many pigs and chickens might there be?

Figure 5.2 is a reformulation of this common problem. It was suggested by one of the teachers in the *Maths in the Kimberly* project as being more suitable for her students.

Figure 5.2

> A ute has some people and some dogs in the back. There are 10 heads and 26 legs.
>
> **Question:** How many people and how many dogs are there?

The problem and the mathematics are the same, but the context is different. Such changes to questions and tasks should be made by teachers to make them appropriate for their students.

A second aspect of this principle relates to using assessment information to inform teaching. Callingham (2010) at the *Teaching Mathematics? Make it count* conference described the important role of assessment and some key processes that teachers can adopt. Similarly Hattie (2009) and Swan (2005) each argued for the constructive use of the students' prior knowledge, and to obtain this teachers will need to assess what their students know and can do. Clarke and Clarke (2004) recommended teachers build connections from prior lessons and experiences and use data effectively to inform learning. Anthony and Walshaw (2009) emphasised building on student experience and thinking. The earlier discussion in the first subsection of Section 3 about insights from students' responses to NAPLAN questions also illustrates ways that teachers can use data to inform their teaching.

Principle 3: Fostering engagement

This principle is elaborated for teachers as follows:

> Engage students by utilising a variety of rich and challenging tasks that allow students time and opportunities to make decisions, and which use a variety of forms of representation.

This principle is fundamentally about seeking to make mathematics learning interesting for students. After reviewing videotapes of a broad range of mathematics lessons, Hollingsworth et al. (2003), suggested that:

> *… students would benefit from more exposure to less repetitive, higher-level problems, more discussion of alternative solutions, and more opportunity to explain their thinking.*

(Hollingsworth et al., 2003, p. xxi)

Hollingsworth et al. also argued that students need:

> *… opportunities … to appreciate connections between mathematical ideas and to understand the mathematics behind the problems they are working on.*

(Hollingsworth et al., 2003, p. xxi)

Swan emphasised appropriate challenges and challenging learning through questioning; Good et al. (1983) recommended the use of higher order questions, Clarke and Clarke (2004) suggested using a range of practical contexts and representations having high expectations, and Anthony and Walshaw (2009) argued it is critical that teachers use 'worthwhile tasks' which is interpreted to mean they are meaningful and relevant to the students. Implementing this principle will present challenges for some mathematics teachers and these strategies can effectively be the focus of teacher learning. Sullivan (2010) in his conference presentation inferred from student surveys that their preferences are diverse and so the breadth of students' interests can only be addressed by teachers effectively presenting a variety of tasks.

Principle 4: Differentiating challenges

This principle is elaborated for teachers as follows:

> Interact with students while they engage in the experiences, encourage students to interact with each other, including asking and answering questions, and specifically plan to support students who need it and challenge those who are ready.

Fundamentally, this principle is about differentiating student support according to the different needs of individual students. It is also about the overall vision of what constitutes an effective classroom dynamic and structure. As will be argued in Section 7, students are more likely to feel included in the work of the class, and to experience success, if teachers offer *enabling prompts* to allow those experiencing difficulty to engage in active experiences related to the initial goal task, rather than, for example, requiring such students to listen to additional explanations, or assuming that they will pursue goals substantially different from the rest of the class. Likewise, those students who understand the task and complete the work quickly can be given *extending prompts* that challenge their thinking, within the context of the original task that was posed. Enabling and extending prompts are elaborated on in Section 7 of this review paper, and examples of the types of tasks, including open-ended tasks, which are most suited to the creation of such prompts are presented in Section 6.

There are other dimensions associated with this principle. Smith (1996) suggested that teachers should predict the reasoning that students are most likely to use, and choose appropriate representations and models that support the development of understandings. Swan also emphasised the notion of community which he linked to positive relationships and to encouraging learners to exchange ideas. Similar ideas emanate from Clarke and Clarke (2004), who emphasise the importance of the teacher holding back and encouraging students to explain their own thinking. Anthony and Walshaw (2009) also emphasised processes for assisting students in making connections.

Principle 4 also connects to *The Shape of the Australian Curriculum: Mathematics*, which has an explicit intention that all students have opportunities to access. It argued:

> *The personal and community advantages of successful mathematics learning can only be realised through successful participation and engagement. Although there are challenges at all years of schooling, participation is most at threat in Years 6–9. Student disengagement at these years could be attributed to the nature of the curriculum, missed opportunities in earlier years, inappropriate learning and teaching processes, and perhaps the students' stages of physical development.*

(ACARA, 2010a, p. 9)

The implication of the ACARA document is that pedagogies need to provide opportunities for all students, especially those who experience difficulty in learning.

Principle 5: Structuring lessons

This principle is elaborated for teachers as follows:

> Adopt pedagogies that foster communication and both individual and group responsibilities, use students' reports to the class as learning opportunities, with teacher summaries of key mathematical ideas.

This principle is essentially advice about the structuring of lessons. There is a lesson format that is commonly recommended to Australian teachers, which in summary is described as: Launch; Explore; Summarise; Review. Yet this rubric does not communicate the subtlety of the ways of working that are intended by this principle. This principle of teaching can be learned from the Japanese way of describing the structure of their lessons. Inoue (2010, p. 6), for example, used four terms: *hatsumon*, *kikanjyuski*, *nerige and matome*, which are described below:

Figure 5.3: The elements and structure of Japanese mathematics lessons

> *Hatsumon* means the posing of the initial problem that will form the basis of the lesson, and the articulation to students of what it is intended that they learn.
>
> *Kikanjyuski* involves individual or group work on the problem. The intention is that all students have the opportunity to work individually so that when there is an opportunity to communicate with other students they have something to say. There is a related aspect to this described as *kikanshido* which describes the teacher thoughtfully walking around the desks giving feedback and making observations that can inform subsequent phases in the lesson.
>
> *Nerige* refers to carefully managed whole class discussion seeking the students' insights. There is an explicit expectation that students, when reporting on their work, communicate with other students.
>
> *Matome* refers to the teacher summary of the key ideas.

The last two steps are the least practised by Australian mathematics teachers, and the Hollingsworth et al. (2003) report on Australian mathematics teaching in the TIMSS video study found them to be very rare. There is an assumption in this Japanese lesson structure, and also in teaching Principles 3 and 4, that students will engage in learning experiences in which they have had opportunity for creative and constructive thinking. This Japanese lesson structure assumes that all students have participated in common activities and shared experiences that are both social and mathematical, and that an element of their learning is connected to opportunities to report the products of their experience to others and to hear their reports as well. Wood (2002) described this as emphasising the interplay between students' developing cognition and:

> *… [the] unfolding structure that underlies mathematics …*

and

> *… rich social interactions with others substantially contribut[ing] to children's opportunities for learning.*

(Wood, 2002, p. 61)

For the mathematical aspects, it is argued that students can benefit from either giving or listening to explanations of strategies or results, and that this can best be done along with the rest of the class with the teacher participating, especially facilitating and emphasising mathematical communication and justification. A key element of this style of teaching and learning is students having the opportunity to see the variability in responses (Watson & Sullivan, 2008), and confirming this variability can indicate underlying concepts for students.

Jill Cheeseman (2003), drawing on the case studies from the ENRP project (Clarke et al., 2002), similarly argued that a lesson review:

> *… involves much more than simply restating the mathematics. It encourages children to reflect on their learning and to explain or describe their strategic thinking. The end of the session gives the opportunity for teaching after children have had some experience with mathematical concept.*

(Cheeseman, 2003, p. 24)

An interesting aspect of the role of language in both teaching and lesson review was described by D. J. Clarke (2010) at the *Teaching Mathematics? Make it count* conference when he connected language, culture and mathematics. He reported a detailed study that compared public and private utterances by teachers and students, noting the variability in usage both within and across cultures. He argued that:

> *… in conceptualising effective learning, researchers, teachers and curriculum developers need to locate proficiency within their framework of valued learning outcomes.*

(D. J. Clarke, 2010, p. 3)

Another aspect of reviews at the end of lessons is the contribution they make to social learning. This is related to a sense of belonging to a classroom community, and is also connected to building awareness of differences between students and acceptance of these differences. Such differences can be a product of the students' prior mathematical experiences, their familiarity with classroom processes (Delpit, 1988), their social, cultural and linguistic backgrounds (Zevenbergen, 2000), the nature of their motivation (Middleton, 1995), persistence and efficacy (Dweck, 2000), and a range of other factors.

Principle 6: Promoting fluency and transfer

This principle is elaborated for teachers as follows:

> Fluency is important, and it can be developed in two ways: by short everyday practice of mental processes; and by practice, reinforcement and prompting transfer of learnt skills.

This principle is familiar to most mathematics teachers, but it is possible to misinterpret the purpose of practice and prompting transfer. Skemp (1986) contrasted mechanical with automatic skills practice. With mechanical practice, students have limited capacity to adapt the learnt skill to other situations. With automatic practice, built on understanding, students can be procedurally fluent while at the same time having conceptual understanding. The advantages of fluency were described by Pegg in 2010 and were analysed in detail in Section 2. Likewise, the importance of prompting mathematical knowledge transfer was clearly argued by Bransford, Brown and Cocking (1999), and the importance of this for learners' future lives was mentioned in Section 2.

Concluding comments

This section presented a synthesis of research recommendations, through its six principles for teaching mathematics, that can be used both individually and collectively. As a group, the set of six principles underpin much of the text about pedagogy and tasks that follows in this review paper. They can and should be used to inform teacher learning, and this aspect of them will be further discussed in Section 9.

section 6

The role of mathematical tasks

Whether in the context of developing practical or specialised mathematics, or in finding ways to encourage the breadth of mathematical actions, or in seeking to engage students in learning mathematics, the key decision that the teacher makes is the choice of task. This section outlines a rationale for the importance of appropriate tasks, illustrates some exemplary types of tasks that have been found to be useful for teachers in facilitating the learning of their students, explains some constraints teachers may experience when using challenging tasks, and describes some students' views on tasks.

Based on extensive research on the impact of mathematical tasks on student learning in the United States of America, a model of task identification and use was presented in a diagram by Stein, Grover and Henningsen (1996), which, when converted to text, proposes the following: the features of the mathematical task as set up in the classroom, and the cognitive demands it makes of students, are informed by the mathematical task as represented in curriculum materials. These are, in turn, influenced by the teacher's goals, subject-matter knowledge and their knowledge of their students. This then informs the mathematical task as experienced by students which creates the potential for their learning.

The teacher determines the learning goals which they hope to have their students achieve and the types of mathematical actions in which the students will engage, noting the levels of student readiness – choosing the appropriate tasks is the next step. It is critical that teachers are mindful of the pedagogies associated with the task, and are ready to implement them. The process of converting tasks to learning opportunities is enhanced when students have opportunities to make decisions about either the strategy for solving the task or the process they will adopt for addressing the task goal or both. In addition, it is expected that the task will provide some degree of challenge, address important mathematical ideas and foster communication and reasoning. It is only tasks with such features that can stimulate students to engage in creating knowledge for themselves.

Why tasks are so important

Many commentators have argued that the decisions teachers make when choosing tasks are critical. Christiansen and Walther (1986) argued that the mathematical tasks that are the focus of classroom work and problem solving determine not only the level of thinking by students, but also the nature of the relationship between the teacher and the students. Similar comments have been made by Hiebert and Wearne (1997), Brousseau (1997), and Ruthven, Laborde, Leach and Tiberghien (2009). In terms of the mathematical actions described by Kilpatrick et al. (2001), and analysed in Section 2 of this review paper, it is not possible to foster adaptive reasoning and strategic competence in students without providing them with tasks that are designed to foster those actions.

Drawing on an extensive program of research on student self-regulation in the United States of America, Ames (1992) argued that teachers can influence students' approach to learning through careful task design. In synthesising task characteristics suggested by other authors, she suggested the main themes were the benefits of posing a diversity of tasks types, presenting tasks that are personally relevant to students, tasks that foster metacognitive development and those that have a social component.

Carole Ames further argued that students may benefit if teachers direct attention explicitly to the longer term goals of deep understanding, linking new knowledge to previous knowledge, as well as to its general usefulness and application. She urged a focusing on the mastery of the content rather than performance to please the teacher or parents, or even students' self-esteem through any competitive advantage. Ames (1992) explained the connection between student motivation, their self concept and their self goals, and argued that it is possible to foster positive student motivation through the provision of tasks for which students see a purpose. The relationship between student motivation and learning is further elaborated in Section 8.

Ames' findings are complemented by suggestions about tasks from Gee (2004), who formulated a set of principles for task design, derived from the analysis of computer games that had proven engaging for children and adolescents. Those of his principles that relate to mathematics task formulation were for:

- learners to take roles as 'active agents' with control over goals and strategies
- tasks to be 'pleasantly frustrating' with sufficient, but not too much challenge
- skills to be developed as strategies for doing something else rather than as goals in themselves.

While it is difficult to identify mathematics classroom tasks that incorporate all of these characteristics, both Gee's and Ames' recommendations provide a suitable standard to which teachers should aspire.

The following subsection describes some different types of mathematics tasks, including those that focus on developing procedural fluency, those that use a model or representation, and those that use authentic contexts. It also describes two types of open-ended tasks, and tasks that progressively increase the complexity of the demand on students. The discussion of the types of tasks is intended to indicate to teachers some options for the tasks they pose, and also the range of types from which they can choose.

Tasks that focus on procedural fluency

The most common tasks in textbooks are those that offer students opportunities to practice skills or procedures, being what Kilpatrick et al. (2001) described as procedural fluency. As argued in Section 2, it is essential that mathematics teaching goes beyond this focus. Yet fluency, across many actions is indeed what students need to be very familiar with, so it is important that tasks that seek to develop fluency are chosen well and incorporated effectively into lessons.

As indicated by Hollingsworth et al. (2003), it is common for mathematics teachers, especially from middle primary years onwards, to demonstrate specific procedures to their students, supplemented by repetitious practice of similarly constructed examples, the intent of which is to develop procedural fluency. This process is both boring and restrictive for students.

It is possible to learn about the processes of choosing good fluency tasks from considering alternative approaches to collaborative planning, commonly undertaken by Japanese teachers. The focus of Japanese mathematics lessons is often on the intensive study of particular examples, with students working on a single task for a whole lesson. This seems to have major advantages for the robustness of the mathematics learning, as is evident in the high standing of Japanese students in international comparative studies.

Fujii (2010) has analysed a range of school texts and research studies focusing on recommendations about what should be the first task posed to young children who are ready to move to subtraction involving numbers beyond 10. Fujii reported that in preparation for lessons with such a focus, Japanese teachers discuss among themselves characteristics of various examples such as whether 13 minus 5 offers more potential than 15 minus 7 for encouraging exploration of key ideas. In the lesson that results, the teachers pose only the task they have chosen. The intention is that students, after working on the task for themselves, will hear a range of strategies for completing such tasks devised by other students and then evaluate their own strategy against other suggested strategies. While this does not align with the ways that Australian teachers commonly utilise tasks that develop procedural fluency, choosing illustrative examples for detailed attention by the group could usefully be a focus of teacher planning. The advantage of doing this is that, rather than mindlessly following rules, students will come to see 'efficiency in strategy' as a matter for their conscious choice.

In considering which tasks will best foster fluency, teachers should also be looking to find optimal ways to incorporate tasks which develop fluency into lessons. Good et al. (1983), in their pivotal review of the teaching effectiveness literature, the importance of which has not lessened over time, described a lesson structure with the following sequence.

- After correcting homework, the teacher poses some old examples to check student facility with prerequisite skills.
- The teacher then presents some new examples, and asks students to complete some illustrative tasks.
- Next, further questions are posed in sets of similar complexity.
- Then the students' responses to set exercises are corrected.
- Some further examples are posed to the class to check both the students' accuracy and their capacity to explain the process they used.
- Further examples are set for homework.

This sequenced structure is more likely to enhance the flow of lessons focusing on developing procedural fluency than the mere setting of examples for practice. The structure also has the potential to enhance conceptual understanding and develop some adaptive reasoning as well.

Tasks using models or representations that engage students

There has been substantial and sustained interest in Australia in tasks and lessons using interesting models or representations that both illustrate key mathematical principles and which have potential to engage students. A web-based teacher resource which is widely used in Australia and elsewhere, *Maths 300* (2010), and which is an extension of the *Mathematics Curriculum and Teaching Program* (Lovitt & Clarke, 1988), presents an outstanding collection of tasks and lessons that use models to represent mathematical or practical situations. Barbara Clarke (2009) described such tasks as 'representational tasks', ones that are:

> *... explicitly-focussed experiences that engage children in developing and consolidating mathematical understanding.*

(Clarke, 2009, p. 178)

The category of tasks that she described were intended to present physical or other representations that made abstract mathematics more tangible for students.

In his work, which built on an extensive program of research and development of tasks at the Shell Centre in the United Kingdom, Malcolm Swan (2005) encouraged teachers to 'use rich collaborative tasks'. According to Swan, these tasks are ones that:

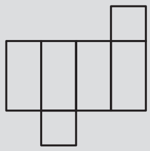
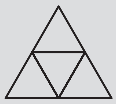
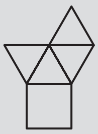
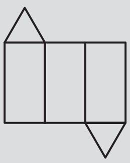
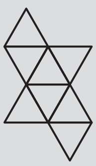
- emphasise methods rather than answers
- facilitate connections between topics

- support cooperative group work
- build on what the students bring to sessions
- explore common misconceptions.

An illustrative task using representations

The following is an example of such a task, adapted by the author from one of Swan's suggestions. This task is suitable for students from about Year 5, and similar examples have appeared on NAPLAN and met with student facility. In this task, the teacher provides students with a shuffled set of cards about one or several different polyhedra. The full set (see Figure 6.1) has cards for five different polyhedra, a total of 20 cards. For each polyhedron, there are four cards to that set – a name card, one with a representation of the net of the specific polyhedron, and two cards that refer to its properties of faces, edges and vertices. The task is for students to identify the polyhedra under discussion, by selecting the four descriptive and representational cards that match that form. The intention is that students will imagine and describe what each polyhedron would look like, but it is also possible to include photographs or actual models of the polyhedra. (See the cover of this review for a depiction of some of the cards.)

Figure 6.1: Polyhedra task cards

I am a rectangular prism	My net is 	I have 6 faces and 8 vertices	I have 12 edges
I am a tetrahedron	My net is 	I have 4 faces and 6 edges	I have 4 vertices
I am square pyramid	My net is 	I have 8 edges and 5 vertices	I have 5 faces
I am a triangular prism	My net is 	I have 6 vertices and 9 edges	I have 5 faces
I am an octahedron	My net is 	I have 8 faces and 6 vertices	I have 12 edges

The presentation of this task in Figure 6.1 is necessarily static and readers are reminded that each square would be cut up into a small card and presented to students for them to assemble and allocate, according to polyhedra form. The pedagogic idea here is that the students will work in small groups to sort the cards with instructions about requirements for them to explain their thinking as they match cards that are different representations or properties of particular polyhedra. Many of Swan's tasks have elements similar to doing a puzzle which can engage students.

Such tasks can assist students in finding, clarifying and using appropriate language, they can provide a focus on different representations of the same idea, and their 'solution' can indicate to teachers what the students know. Such tasks are ideal for building conceptual understanding, and are readily adaptable to working in the same way in many other content domains in mathematics.

Contextualised practical problems

The use of contexts to situate mathematical problems is common internationally. Raffaella Borasi (1986), for example, defined 'context' as the situation in which a problem is embedded, providing problem solvers with information that may assist them in solving the problem. Meyer, Dekker and Querelle (2001), building on Borasi's notion, suggested that contexts can be used to motivate, can illustrate potential applications, can be a source of opportunities for mathematical reasoning and thinking, and can anchor student understanding.

In studying the classroom work samples and test responses of 273 children in six Year 4 classrooms and six Year 6 classrooms in the United States of America, Wiest (2001) found that the context of problems affected learning. A range of variables affected included students' interest in, and therefore their attentiveness and willingness to engage with problems; the strategies they used; the effort they expended; their perception of the difficulty of the task and their success in solving it; and the extent to which measurable learning outcomes were attained. In its policy directions report on curriculum and evaluation standards the *National Council of Teachers of Mathematics* (1989), the peak professional body representing mathematics teachers in the United States of America, argued that problems using contexts enrich the experience of students learning mathematics. Brinker-Kent (2000) studied the use of mathematical tasks, set in contexts that were meaningful to the students in a culturally diverse elementary school, and concluded that all students are capable of learning significant concepts when they have the opportunity to explore the ideas in contexts that are meaningful to them.

In reporting on a study of teacher development in, and the classroom implementation of, a range of types of tasks, 'contextualised practical problems' were described by Clarke and Roche (2009) as occurring when the teacher situates mathematics within a realistic context to engage the students, with the motive of using the context as a stimulus for learning the mathematics. Many of the *Maths 300* (2010) tasks and lessons fall into the category of contextualised tasks.

The following task was developed by Doug Clarke and Anne Roche as part of an interactive student assessment intended for students in the upper primary and junior secondary years. They used attractive images of realistic cards, although the task is presented here as text.

Figure 6.2

> If one pre-paid card for downloading music offers 16 songs for $24, and another offers 12 songs for $20, which is the better buy?

Tasks such as this one address both practical and specialised mathematical goals. The task is practical in that using pre-paid cards to purchase is a realistic context for students and so the context and the task would be familiar to them. The task also addresses an important application of ratios and rates (that of 'best buys'). There is a diversity of mathematical strategies that can be used to solve this task. These diverse strategies range from unitary methods (either comparing the number of songs per dollar, or cost per song), the common comparison methods (either the number of songs for $120, or cost of 48 songs), comparison of change method (4 extra songs for $4 more), and so on. The pedagogic point is that each of these strategies allows the teacher an opportunity to name and emphasise specialised mathematical ideas, once they have arisen from the students' investigations. Such tasks offer substantial potential to develop strategic competence and adaptive reasoning, are engaging for students, and are suited to collaborative activity.

Open-ended tasks

Various researchers have found that dealing with tasks or problems that have many possible solutions contributes to learning. Such researchers include those working on investigations (Wiliam, 1998), problem fields (Pehkonen, 1997), and the open approach (Nohda & Emori, 1997). Christiansen and Walther (1986) argued that tasks with open goals (that is many possible solutions) can engage students in productive exploration, and Middleton (1995) proposed that such tasks enhance motivation through increasing the students' sense of control. There are many types of open-ended tasks and the following elaborates just two types: investigations; and content specific tasks.

Investigations

The following example of an investigation type of task is an adaption, by the author of this review paper, of work by some of the researchers referenced above.

Figure 6.3

> Collect some sports balls, such as a basketball, a baseball, a table tennis ball, and tennis ball. Describe these balls.

The intent of this task is that students will define the properties (such as dimensions, mass, texture) of the balls on which they will focus, and then find ways to both describe the individual balls and compare the characteristics of the balls. It requires students to make choices, describe, measure, record, explain, and justify, which constitute some of the desired mathematical actions described in Section 2.

This sport balls task (Figure 6.3) is also similar to the 'rich tasks' proposed as cross-disciplinary investigations by the Department of Education and Training, Queensland (2011). The following is a description of the learning involved in such a rich task, labelled *Pi in the sky*. It was explained for teachers as follows:

> *Students will demonstrate an understanding of different mathematical approaches used to frame and answer questions about astronomy asked by cultures from three different historical ages. For each culture, they will immerse themselves in one such question as well as the ways in which the culture used or developed mathematics to frame and answer the question.*

(Department of Education and Training, Queensland, 2011, p. 2)

While such tasks present a kaleidoscope of options, which have potential to enable considerable creative learning, they are also extremely difficult for teachers to implement and for students to navigate. There are a number of challenges for teachers and students in seeking to solve such investigative tasks. First, there is substantial extraneous information that must be processed. Second, because such tasks have several different mathematical elements, it is hard for teachers to align such tasks with a curriculum that is sequential and already crowded. Third, it is difficult for students to know what they are meant to be learning. Finally, the lack of clarity in the focus makes the task of teaching difficult. While there are reports of teachers implementing such tasks effectively, often in a cross-disciplinary collegial culture, understanding and implementing such tasks also present challenges, which explains their limited uptake by teachers.

Content specific open-ended tasks

A similar approach, one that retains most of the benefits associated with such investigations, but which is more manageable for students and teachers, is one that involves open-ended tasks which are aligned with a sequential and topic specific curriculum. One way to do this is with what are described as content specific open-ended tasks. Some examples of such tasks are as follows:

Figure 6.4

a Draw some letters of the alphabet on squared paper so that each letter has an area of 10 square units.

b A commercial vegetable garden which has the shape of an 'L' has an area of 1 hectare. What might be the perimeter?

c On squared paper, draw as many different parallelograms as you can with an area of 12 square units.

Each of these tasks addresses a specific aspect of the curriculum, ranging respectively from calculating area as counting squares to forming composite shapes, to developing a generalised rule for calculating area. Each has multiple possible solutions and solution strategies and will therefore encourage and allow rich classroom discussions and create the expectation that students will explain what they have done and their underlying thinking.

For example, in the case of the second task, because it does not need to be solved by the application of a taught routine, the students can make choices on the size of the parts of the 'L', and therefore expect to be invited to explain those choices. Those explanations provide teachers with important insights into the thinking of their students. The tasks are accessible for most students since they can respond to the task by building on their current knowledge and understanding. In the first example above, some students might respond by drawing simple letters (L, C, T), whereas others might use more complicated letters involving many half squares (Z, K, R). The tasks are engaging for students because they choose the level at which they engage. The tasks promote different types of thinking because there are many solution strategies possible, and different ways of thinking, since responses can be represented in different ways.

There are a number of pedagogic matters to be considered in relation to teaching open-ended tasks. One pedagogic advantage of such tasks is that they are readily adaptable for students who experience difficulty, and can easily be extended for students who finish quickly, as will be discussed in Section 7. Another advantage is that devising them is facilitated by collegial action between teachers. A further issue is that the classroom climate needs to be managed so that students feel free to offer their ideas on possible options. This matter will be further discussed later in this section.

To further illustrate the ways that such tasks might contribute to learning, the following discussion deals with issues which often arise, especially when working with students in the junior secondary years, on Figure 6.4c (drawing different parallelograms of a given area). In classroom trials, students working on this task have been prompted to query their conceptual understandings. They commonly ask questions about:

- whether rectangles and squares are also parallelograms. This leads to a class discussion of the inclusiveness and parsimony of those definitions.
- which parallelograms can be treated as the same and which as different (that is, whether a parallelogram with a base of 4 and a height of 3 is the same as one with a base of 3 and a height of 4, if their angles are the same). This query allows discussion of the concepts of transformations, and congruence.
- whether there are only two parallelograms with a base of 4 and a height of 3. Most students are able to find the shape with a base of 4 and with the side going along the diagonal of the squares on the paper, and, with prompting, also find the rectangle. But many have difficulty finding others. This allows discussion of the key issue that the area of the parallelogram is determined by the base and height and not by the interior angles or the length of the side. Therefore, there *can* be multiple such parallelograms!

These concepts are important mathematical ideas, itemised in many Australian mathematics curricula. By using open-ended tasks, teachers can facilitate discussion of such ideas, thus giving students an opportunity to develop conceptual clarification and proficiency in mathematical reasoning. This pedagogy models adaptive application of mathematical concepts and processes which, as argued in Section 4, will be important to these students in their learning. Each of these key mathematical issues arises as an outcome of the students' explorations, and so such tasks allow teachers to emphasise all five mathematical actions described by Kilpatrick et al. (2001), and analysed in Section 2 of this review paper.

Constraints on use of tasks

A first step in addressing constraints is awareness of them. One of the major constraints that teachers experience when utilising such tasks is that many students avoid risk taking and do not persist with the challenges that are required in order to complete the task. And teachers are sometimes complicit in this avoidance strategy.

Desforges and Cockburn (1987), for example, reported on a detailed study of primary classrooms in the United Kingdom and found that students and teachers conspired with each other to reduce the level of risk for the students. Desforges and Cockburn argued that teachers can sometimes avoid the challenge of dealing with students who have given up, by reducing the demand of the task. Stein et al. (1996), in a detailed classroom-based study of task implementation, also noted the tendency of teachers to reduce the level of potential demand of tasks for some students. Tzur (2008) has argued that teachers sometimes modify tasks at the planning stage if they anticipate that students cannot engage with the tasks without considerable assistance, and also once they see students not responding as intended (see also Charalambous, 2008). It is important for teachers to be aware of this tendency and, if they note it in themselves, to develop strategies to overcome this tendency.

After recognising the role of the teacher in creating an optimal learning environment for all students, the next step is establishing a classroom culture that builds community, encourages effort and acceptance of errors, and not only tolerates, but celebrates, difference. Such a classroom culture can be established through explicit norms.

Cobb and McClain (1999) used the term 'mathematical norms' to describe mathematical tasks and their possible trajectories, the mathematical actions which are to be valued in the learning, and all and any of the products which students contribute to the learning in classes. Complementing this notion of mathematical norms is the concept of socio-mathematical norms, which includes the modes of communication, types of responses valued, and expectations about risk taking and tolerance of others' errors. One of the roles of the teacher is to establish the norms that operate in the classroom so that the type of task use described in this section is not subverted deliberately or inadvertently by the actions of some students. A necessary prerequisite to implementing the type of teaching based on representational, contextual and open-ended tasks described above is the establishment of the appropriate classroom culture.

Problem posing

One strategy that may be useful for encouraging students to engage with problems and to persist, even if challenged, is described as 'problem-posing'. Leung (1998) and English (2006) both proposed problem posing as a process in which students not only reformulate problems with which they are presented, but they argued students should also pose their own problems, either for themselves or for others in the class to solve. One effect is that students generally pose problems at the level with which they are most comfortable, and the teacher's challenge is then one of how to move them into a less comfortable stage by proposing more complex tasks, possibly with other students. Another effect of problem posing is that many of the above tasks require students to ask themselves questions, even when they are working on a specific task, and so a willingness to ask questions about what is possible assists students in exploring the potential of many tasks. This capacity to pose questions is one of the goals of mathematics teaching, and it has uses in adult life too. It is the essence of what Kilpatrick et al. (2001) described as 'adaptive reasoning', and has important elements of strategic competence.

Seeking students' opinions about tasks

In this author's presentation at the *Teaching Mathematics? Make it count* conference, he reported on a project that sought insights into aspects of task use by examining student preferences (Sullivan, 2010). The project drew on earlier research on students' attitudes (McLeod & Adams, 1989) and their beliefs about mathematics and its learning (Leder, Pehkonen & Törner, 2002; Pajares, 1992), and the value they ascribe to learning mathematics (Bishop, 2001). Sullivan outlined how he, with colleagues D. Clarke and B. Clarke, surveyed students from 95 Year 5 to Year 8 classes, inviting them to compare different types of tasks and to indicate their preferences for the tasks they liked doing and also those from which they felt they best learn.

The researchers found that students have a wide diversity of preferences for the types of tasks that they enjoy and also for the types of tasks from which they think they can best learn. Most significantly, the students' preferences for particular types of tasks were not dependent on whether they were confident in their own ability or whether they reported positive attitudes to learning mathematics. Sullivan (2010) concluded that the diversity of student preferences make it essential that teachers incorporate a variety of task types into their planning and teaching, so that they 'reach' all their students, and also that they explain the purpose of each type of task to the students, so students can more readily recognise the type of task it is.

Concluding comments

This section has argued that choosing appropriate tasks is one of the key decisions for teachers, and has presented a range of possible types of tasks, all of which can make a positive contribution to different aspects of student learning. It has been established also that having students pose questions in their own words allows for explicit articulation of mathematical learning, and for the understanding that there may be multiple ways of solving a problem. Recognising the value of these mathematical learning outcomes, both during the school years and in later life, research suggests that mathematics teachers should incorporate most of these types of tasks into the planning of lesson sequences, and during individual lessons. Indeed, one of the expectations in *The Shape of the Australian Curriculum: Mathematics* (ACARA, 2010a) is that teachers will use a range of types of tasks that allow students opportunities to solve problems, explain their reasoning, and build their understanding, as well as developing the necessary fluency.

Of course using challenging tasks creates its own expectations for teachers, including the need to maintain the demand of tasks and to support students in persisting, as is further discussed in the next section. All of these aspects of task choice and use can productively be the focus of teacher learning initiatives, as discussed in Section 9.

section 7

Dealing with differences in readiness

Differences in readiness within most Australia classes are significant. In its principles for the development of the Australian Curriculum, ACARA (2010b) noted that the top 10 per cent of students are typically five years ahead of the bottom 10 per cent. This situation is a world-wide occurrence, with, for instance, the Cockcroft report (1984) describing a seven-year range in achievement in Year 9 classes in the United Kingdom. Planning for and managing this difference in readiness is the biggest challenge for teachers in all schools.

Mathematics teachers, arguably more than most teachers, find in every lesson that they must address the challenge that some students learn the current content quickly, while others require substantial support. Section 2 provided data which demonstrated the very substantial range in achievement that exists in most mathematics classrooms. Section 6 outlined a range of types of tasks that teachers can use to foster the desired mathematical actions, and indicated ways in which these tasks can be extended. This section will analyse the nature of the challenge of teaching classes with a potential knowledge range of at least five school years, first by critiquing the common approach of dealing with differences in readiness through grouping students into like ability groups. This review paper will then outline an alternative approach to dealing with the student differences in knowledge, that of building an effective classroom community, and by differentiating the demand of tasks.

The challenges that teachers experience

The challenge for teachers in addressing the diversity of readiness of students was the explicit focus of research involving lesson observations undertaken by Sullivan, Mousley and Zevenbergen (2006). They reported on many upper primary lessons, in a wide range of Victorian schools serving communities from different diverse socioeconomic, regional and cultural backgrounds. They observed many teachers who were highly competent and who had planned lessons that used interesting contexts and engaging tasks, which allowed students scope to make active decisions, to select their own approaches to solving the tasks and to choose their own methods of representing their solutions. The teacher's intention in many of the observed lessons was for the students to learn by solving problems through their own thinking, under guidance from, as distinct from following the direction of, the teacher.

Sullivan et al. (2006) reported that teachers confronted three main challenges. The first challenge was connected to the posing of the interesting tasks. Because students were required to think for themselves over multi-step tasks, there were occasions when many students who had decided the task was too difficult for them and so behaved as though they were disengaged. Even though most of these teachers fostered a classroom culture in which effort and persistence were valued, there were many occasions when teachers responded to the apparent disengagement by stopping the students and having a general discussion on barriers that the students may have experienced, usually drawing on insights about the task from the students. So the classes were more teacher-focused than had been planned or intended.

The second main challenge was that, irrespective of the success of the whole class discussion, often a few students needed further prompts or other support that could allow them to re-engage with the task. While the first type of challenge could be productively elaborated for the whole class, it was not helpful for the teacher to address the second type challenge with the whole class. The difficulties experienced were specific to particular students and so little purpose would be served by further whole class interactions which could have been potentially counter-productive. This presented management difficulties, and the learning for some students was less than full potential.

The third main challenge is that some students completed the required tasks quickly, creating pressure on the teacher to move onto new content or tasks, or resulting in some disorder. Teachers need to, and can, address this third challenge which is similar to the second challenge, by working with students individually. However, the nature of the teacher prompts is quite different.

Both the second and third challenges occur in more or less every class, and where possible it is more productive for the teacher to interact with individuals as discussed later in this section. Referring to the differences in achievement evident in Table 3.1, students not achieving level 2 would experience difficulty dealing with most content at Year 9 level. Yet those achieving at the highest levels would complete the work quickly and be ready for further challenges. Indeed, even if the students are grouped together in like achievement groups, as is discussed in the next subsection, teachers will always face the challenge that some students need additional support while others complete the work quickly.

Impact of grouping students by achievement

Many schools address the issue of diversity in achievement by grouping students by achievement, either for all or some of their subjects, often according to their performance *in* mathematics, and commonly *for* mathematics. This procedure, described by Hattie as 'tracking' (the term 'streaming' is also often used in Australia), has been widely criticised, perhaps most compellingly by Hattie in his review of over 300 research studies of tracking. He concluded that:

> *… the results show that tracking has minimal effect on learning outcomes and profound negative equity effects.*

(Hattie, 2009, p. 90)

Zevenbergen (2003) has also argued that the most commonly observed effect of streaming is reduced opportunities for students in the lower groups, which is what Hattie meant by the term 'equity effects'.

Self-fulfilling prophecy and self-efficacy effects

In his research decades ago, Brophy (1983) writing in the context of mathematics, argued that one reason for the counter-productive impact of ability grouping on low-achieving students, across any field of learning, is the negative effect it has on teacher expectations and what he described as 'reduced opportunities' flowing from self-fulfilling prophecy effects. Basically the notion of the self-fulfilling prophecy is that for classes, groups, or individuals, if teachers *think* students can learn (independent of whether they are or not), the students achieve well and if teachers think that particular students *will experience difficulty in learning*, then those students do so. Brophy (1983) posed a cyclic model that describes how self-fulfilling prophecy operates.

1 *Early in the year, teachers form differential expectations for student performance.*
2 *Consistent with these differential expectations, teachers behave differently toward different students.*
3 *This differential teacher behavior communicates to each individual student something about how he or she is expected to behave in the classroom and perform on academic tasks.*
4 *If teacher treatment is consistent over time, and if students do not actively resist or change it, it will likely affect student self-concept, achievement motivation, level of aspiration, classroom conduct, and interactions with the teacher.*
5 *These effects generally will complement and reinforce the teacher's expectations, so that students will conform to these expectations more than they might have otherwise.*
6 *Ultimately, this will make a difference in student achievement and other outcomes, indicating that teacher expectations can function as self-fulfilling prophecies.*

(Brophy, 1983, pp. 639–640)

This description illustrates how teachers may inadvertently influence the achievement of their students if they are not aware of the potential of the expectations they form about students. Similarly, if the teacher feels that students in a low-achieving group cannot solve multi-step problems and so does not pose them, the students will not learn how to solve them.

A further possible explanation for the negative effects of like-achievement grouping is teacher *self-efficacy*, which is the extent to which teachers believe they have the capacity to influence student performance, regardless of school circumstances and student background (Tschannen-Moran, Hoy & Hoy, 1998).

The argument here is that if the teacher believes a particular student (or group of students) experiencing difficulty can learn, then the teacher will pose additional tasks, offer alternative explanations, try different representations or vary the context. If, on the other hand, a teacher does not believe that this student can learn the content, then the teacher might allow the student to avoid engaging with the task, over-explain the solution process, or pose some completely different, less challenging activity. In all such self-fulfilling prophecy situations it is clear that students will not learn the expected content.

Teachers should also be aware of various motivational drivers that students are experiencing, and recognise that teaching is not just the effective presentation of content, but a means of ensuring that students have a positive self concept that will allow them to persist long enough to overcome at least the immediate challenge they confront. Students need to believe in their teachers' capacity to teach them.

A pedagogical model for coping with differences

The proposition of this alternative pedagogical model is that, rather than grouping students by their achievement, teachers should plan for commonality of experiences among class members, with differentiation of the tasks directed toward maximising the chances that all students will have the same basic experience, although they may have this at different times. The proposed pedagogic strategy has been intimated in Sections 5 and 6. The following text outlines three key aspects of this proposition. It describes what differentiation might mean, it outlines a particular research program from which the proposition emanates, and it details the key pedagogical strategies for dealing with differences in readiness, *enabling* and *extending prompts*.

Differentiation

While the term 'differentiation' has been used to describe students experiencing a different curriculum (Bräandsträom, 2000), the approach described in the following is intended to allow for the possibility that the curriculum and overall experience is comparable for all students, but the tasks in which they engage are differentiated.

One of the more powerful notions that can guide teachers' thinking with respect to differentiation is zone of proximal development (ZPD) which is described as the:

> … *distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined by problem solving under adult guidance or in collaboration with more capable peers.*

(Vygotsky, 1978, p. 86)

An important implication is that ZPD defines the work of the class as going beyond tasks or problems that students can solve independently, so that the students are working on challenges for which they need support. Of course, if the teacher poses problems that are challenges for all students and within the ZPD of most, there will be some students who are not yet at the level of independent problem solving for this particular problem or task, even with 'adult guidance'. This implication is that support can be offered to such students through *differentiating* the task on which they are working.

This notion of *task differentiation* is a consistent theme in advice to teachers. The Association of Teachers of Mathematics (ATM) (1988), representing teachers in the United Kingdom, produced a handbook that described in some detail 34 different strategies that teachers might use when intervening while students are working. The strategies are grouped under headings that list major decisions such as: whether or not to intervene; how to initiate an intervention; and whether to withdraw or proceed with the intervention.

A key element was described as 'active monitoring' by the teacher, which includes decisions by the teacher about when and whether to intervene. Such decisions are informed by: the level of activity of the student(s); the body language of the student(s); and questions the students ask. Having made a decision to intervene, the teacher makes what the ATM called an 'opening gambit', meaning the first, exploratory comment or question. Some examples of an opening gambit are:

- inviting a student to explain what they have done
- inviting a student to explain what they understand the task to be
- posing an alternative task that can assist the student to overcome an impasse or barrier they may be experiencing
- making an explicit suggestion
- correcting an error or a misconception
- and/or recommending a way to proceed.

The ATM model proposed 14 specific suggestions about the posing of alternative tasks, following the identification of the students' problem by an opening gambit. Of those directed at interventions to support students experiencing difficulty, about half relate to task differentiation.

Similarly, in arguing for recognition of difference in readiness, Christiansen and Walther (1986) argued that:

> *One of the many aims of the teacher is to differentiate according to the different needs for support but to ensure that all learners recognise that these processes of actions are created deliberately and with specific purposes.*

(Christiansen & Walther, 1986, p. 261)

In discussing tasks based on interactive computer games, Gee (2004) suggested that optimal tasks are those that are able to be customised to match the readiness of learners both for those who experience difficulty and for those for whom the core task is not challenging. There is substantial support for the notion of differentiating tasks to accommodate the needs of particular learners, although this occurs within a broader conceptualisation of classrooms as is described in the following. Either way, a teacher's skills in task differentiation are crucial to an effective pedagogy, in mathematics as in all subjects.

A planning model

A planning model that can be used to inform teaching that incorporates differentiation was developed by Sullivan and colleagues (Sullivan, Mousley & Zevenbergen, 2006). Their model was based on research that involved the gradual development, trialling, evaluation and adjustment of the planning model. The initial stage of the research identified and described aspects of classroom teaching that may act as barriers to mathematics learning for some students, and drew on responses from focus groups of teachers and academics to suggest strategies for overcoming such barriers (Sullivan, Zevenbergen & Mousley, 2002). Subsequent research analysed some partially scripted learning experiences taught by participating teachers (Sullivan, Mousley & Zevenbergen, 2004). This analysis allowed a re-consideration of the emphasis and priority of respective aspects of the initial model.

Sullivan et al. (2006) reported that it was possible for teachers to create sets of mathematics learning experiences that resulted in most students being included in rich, challenging mathematical learning. Sullivan et al. then developed a model comprising five key aspects of planning and teaching mathematics. Three of these aspects are described here, and the fourth and fifth are elaborated in the subsection: Enabling and extending prompts.

The tasks and their sequence

As discussed in Section 5, thoughtfully crafted tasks create opportunities for personal constructive mathematical activity by students. A further important aspect of the Sullivan et al. (2006) model is that careful *sequencing* of tasks can contribute to learning. This relates closely to what Simon (1995) called a 'hypothetical learning trajectory' described as:

> *… provides the teacher with a rationale for choosing a particular instructional design; thus, I (as a teacher) make my design decisions based on my best guess of how learning might proceed. This can be seen in the thinking and planning that preceded my instructional interventions … as well as the spontaneous decisions that I make in response to students' thinking.*

(Simon, 1995, pp. 135–136)

Simon noted that such a trajectory is made up of three components. They are: the learning goal that determines the desired direction of teaching and learning, the activities to be undertaken by the teacher and students, and a hypothetical cognitive process, which is described as:

> *… a prediction of how the students' thinking and understanding will evolve in the context of the learning activities.*

(Simon, 1995, p. 136)

The implication is that it is not enough to create interesting tasks: those tasks must be appropriately sequenced.

Explicit pedagogies

The second aspect of the planning model developed by Sullivan et al. (2006), is for teachers to decide on the pedagogies, organisational routines and modes of communication that are connected with the intended student experiences. These decisions, which should be explicitly outlined to the students for each lesson, should incorporate the intended ways of working and reasons for adopting these, the types of responses valued, the teacher's views about legitimacy of knowledge produced, and responsibilities of individual learners.

Sullivan et al. (2002) described how making both content and pedagogical expectations explicit enables a wide range of students to work purposefully, with teachers commenting positively about relatively low levels of teacher–student friction. For example, some of the expectations that can be made explicit include the nature of the communication in framing the problem, the necessary listening skills, matching words with practical situations, and knowledge associated with familiarity with the context.

Making the pedagogies and other schooling practices *explicit* relates to the researchers' agreement with Bernstein's view (1996), that through different methods of teaching, students receive different messages about the overt and the hidden curriculum. They argued:

> *… 'making explicit' needs to take two forms. First, teachers need to become more aware of specific, common aspects of teaching that may not be optimal for certain groups of pupils, and then address these when working at improving their typical patterns of interaction in mathematics classrooms. Second, aspects or approaches to teaching that they decide to use purposefully need to be made more explicit to the children so that potential for confusion is reduced and reasons for using particular strategies are well understood.*

(Sullivan et al., 2002, p. 655)

The intention is that students receive explicit messages about the goals of schooling, the goals of learning mathematics, and the language and culture of school expectations. It contributes to the students feeling they are part of the learning process.

Learning community

A third aspect of the model is the deliberate intention that *all* students progress through learning experiences in ways that allow them to feel part of the class community, to contribute to it – including being able to participate in reviews and summative class discussions about the work undertaken (see also Brown & Renshaw, 2006). The model is underpinned by the view that *all* students will benefit from participation in core activities that form the basis of common discussions and shared experience, both social and mathematical. And it is essential for there to be a common basis for any subsequent lessons and assessment items on the same topic. A further assumption is that there will be a positive classroom culture developed in which the norms of communication reduce the chances that students may be criticised by other students if they make mistakes. Sullivan et al. (2004) reported that the use of tasks and prompts that support the participation of all students resulted in classroom interactions that had a sense of a learning community, with wide-ranging participation in learning activities, as well as group and whole class discussions in which students learn from each other as well as from the task and the teacher.

These three aspects of the planning model, meaning the tasks, the explicit pedagogies and the classroom community, are prerequisites for the key strategies for dealing with difference: enabling prompts and extended prompts.

Enabling and extending prompts

Sullivan et al. (2006) described two kinds of prompts, as strategies that can be directed at students either when they need to be more supported or when they are ready to move forward in their learning. Both kinds of prompts follow the initial task activity. They are 'enabling prompts', described as:

> *… supports offered to student experience difficulty along the way*

and which are:

> *… active experiences related to the initial goals task, rather than for example, requiring such students to listen to additional explanations.*

(Sullivan et al., 2006, p. 123)

Sullivan et al. also described 'extending prompts', that is 'supplementary tasks or questions that extend their thinking on that task' (p. 124) for students who move quickly through the initial tasks. Since teachers can be sure they will have both kinds of students in their class, Sullivan et al. urge teachers to plan ways to have these prompts ready to use whenever students demonstrate a need for such extension.

Enabling prompts

The enabling prompts are tasks that are similar to those undertaken by other class members, but which are differentiated in some way to increase the accessibility of the task for those students who need them. The overall objective of the learning experience and the nature of the associated mathematical activity must be preserved. With such an approach, the whole class completes the same basic task so all students can participate in discussion and class reviews. Most importantly, students are ready to move together onto the next stage of the learning. A key advantage is that the teacher can maintain a sense of the class as a coherent learning community, even though some tasks have been differentiated for individual students.

Enabling prompts involve slightly lowering an aspect of the task demand, such as the form of representation, the size of the number, or the number of steps, so that a student experiencing difficulties can proceed at that new level; and then if successful can proceed with the original task. (This approach was mentioned in Sections 5 and 6.)

This approach of enabling prompts can be contrasted with the more common requirement that such students (a) listen to additional explanations; or (b) pursue goals substantially different from the rest of the class. Sullivan et al. (2004) reported that the use of enabling prompts generally resulted in those students experiencing difficulties being able to start (or restart) work at their own level of understanding, and they were enabled to overcome the barriers they had previously met in the lesson.

Extending prompts

Of course, it is not only students experiencing difficulty who create challenges for teachers, but also students who finish the initial work quickly. These students need to keep learning, and it is the teacher's job to ensure that they do. Indeed, many of the proponents of 'streaming' advocate it precisely because they are concerned that students with an aptitude for mathematics may be held back in heterogeneous classrooms.

To anticipate the needs of students who finish the set work quickly, Sullivan et al. (2006) recommended that teachers plan prompts that *extend* the thinking of students which they can pose to students who complete tasks readily. The prompts need to work in ways that do not make the students feel that they are merely getting 'more of the same'. Students who complete the planned initial tasks quickly receive supplementary tasks or are posed questions that extend their thinking and activity. Extending prompts have proved effective in ensuring that higher-achieving students are profitably engaged and their development is supported by posing higher-level problems. For example, students can be challenged to seek generalisable understandings that are associated with higher order learning. These extending prompts involve more sophisticated thinking on the task being considered by others in the class.

Sullivan et al. (2006) also offered advice to teachers on ways to vary the task, based on the perceived cognitive demands of the task or the cognitive readiness of the student. They recommended that the teacher, for example, make inferences on the cognitive demand of tasks, and the characteristics that may be contributing to the complexity, such as:

> *… [the] number of steps involved, the modes of communicating responses, the degree of abstraction or visualisation required, and even just the size of the number to be manipulated.*

(Sullivan et al., 2006, p. 124)

Each of these enabling prompts involve the reduction of one or more aspects of the demands made by the task, while extending prompts require teachers to extend the experience by deepening the requirement for thinking.

To illustrate the nature of enabling and extending prompts, assume that a teacher had prepared a lesson based on the following task. Assume also that the task is at the appropriate level of challenge for most of the students in the class.

Figure 7.1

> Five people went fishing. The mean number of fish caught was 3 and the median number of fish caught was 2.
>
> **Question:** How many fish might each person have caught?

Some examples of enabling prompts that could be posed to individual students, depending on the nature of the difficulty they are experiencing, are the following.

- Here are *some* 'fish' and *some* 'people'. (The intention is to offer a physical representation of the problem that reduces the problem complexity by one step, but does not reduce the need for students to solve the problem for themselves.)
- Work on this problem: 'Five people went fishing. The mean number of fish caught was 3. How many fish might each person have caught?'
- Work on this problem: 'Five people went fishing. Together they caught 15 fish. How many fish might each person have caught?'

These three enabling prompts, in different ways, reduce the demand of the task and so make it more accessible for those students who could not do the original task.

Some examples of extending prompts that could be posed to students who have finished quickly are the following:

- How many different answers are possible?
- What if the mode number of the fish caught was 1?
- If only four people were fishing, what difference would that make to the mean?

These three extending prompts extend the students' experience in meaningful ways, staying within the context of the original task, but encouraging the students to thinking more deeply about the mathematics.

In other words, one approach to making mathematics teaching more inclusive is for the teacher to treat the class as a learning community, making the pedagogical process explicit. The teacher poses tasks that are challenging for the class but plans enabling prompts that assist those students experiencing difficulty to engage with a variation to the original task, with the intention that they will work on the original task subsequently. The teacher also plans extending prompts to further challenge those students who have completed the original task.

Concluding comments

This section has recognised that wide differences in achievement in most classes create challenges for teachers and for learners also. Grouping students by achievement needs to be avoided, but whatever the mode of grouping students, teachers should be aware of the impact of their expectations of students and of their capacity to effect positive outcomes. The basic proposition of this section was that, rather than grouping students by their achievement, teachers can seek to foster a collaborative classroom community, make pedagogies explicit, thoughtfully sequence tasks, pose enabling prompts for students experiencing difficulty and extending prompts for students who have completed the tasks. To plan to adopt these strategies is responsible pedagogy and requires exceptional skill. Implementing such strategies in open classrooms, where students are free to engage at their own level but are also challenged, requires even greater skill and intent. Professional learning for teachers to develop these skills is the focus of Section 9.

section 8

Ensuring mathematical opportunities for all students

Section 2 included discussion of results of PISA and TIMSS which indicated that overall Australia is performing well in these international comparisons, and there are still many students who continue to study mathematics in the final year of secondary schooling. Yet, one of the consistent findings from those international comparisons and from consideration of Australian assessment data is that there is substantial diversity of student achievement that teachers experience in Australian mathematics classes. This is perhaps the single most important challenge confronting teachers of mathematics at all levels of schooling, as it impacts on the opportunity of all class members to learn effectively.

This section will describe and analyse:

- the impact that inappropriate motivation has in limiting engagement of low-achieving students
- the active role the teacher has in addressing the learning needs of low-achieving students
- specific intervention programs that involve withdrawing students from class for individual or small group tuition that have been found to be effective
- recommended approaches to teaching Indigenous students that illustrate how specific actions may be needed to address the learning needs of particular groups of students.

The goal in this section is to identify the positive aspects of particular approaches and to indicate emphases in initiatives which research has demonstrated contribute to accelerating the learning of low-achieving students.

The underlying theme of the section is to improve the equity of learning outcomes through enhancing the education of low-achieving students. It is emphasised though that the intent is to improve the education of all students. Those students who are progressing normally through the education process, or indeed who have particular aptitude, are themselves disadvantaged when some students are substantially behind the rest of the class. Either the teachers spend disproportionate time with the low-achieving students, and so have less time for the others, or the low-achieving students are disruptive, which also means the teachers spend less time with the others. The value of equal opportunities for all is affirmed by this review paper and is an underlying assumption in the subsections that follow.

Addressing the motivation of low-achieving students

In seeking to gain insights into factors contributing to the underachievement of particular groups of students, it is important to understand factors influencing students' motivation. It appears that one of the major constraints to improving the achievement of low-achieving students is that after a few years of schooling, they believe that they cannot learn mathematics. There has been significant research (Martin & Marsh, 2006) that indicates that self concept is a key determinant of learning outcomes for all students, and that negative self concept is especially deleterious for low-achieving students. Carol Dweck (2000) argued that finding ways to support low-achieving students is as much connected to their orientation to learning as it is to cognitive approaches. Dweck categorised students' orientation to learning in terms of whether they hold either *mastery* goals or *performance* goals.

Students with mastery goals, according to Dweck, seek to understand the content, and evaluate their success by whether they feel they can use and transfer their knowledge. They tend to have a resilient response to failure and they remain focused on mastering skills and knowledge even when challenged. Such students do not see failure as an indictment on themselves, and believe that effort leads to success. In contrast, students with performance goals are interested predominantly in whether they can perform assigned tasks correctly. Such students seek success, but mainly on tasks with which they are familiar. They avoid or give up quickly on challenging tasks, they derive their perception of ability from the extent to which they attract positive recognition, and they feel threats to self-worth when effort does not lead to recognition.

Dweck connected each of these two sets of goals to particular views of intelligence. Performance goals were connected to a fixed view of intelligence called 'entity theory' in which people believe the intelligence that they have is what they were born with and cannot be changed. Dweck suggested that people who believe in the entity view of intelligence require successes to maintain motivation, and see challenges as threats. People with mastery goals, on the other hand, see intelligence as malleable or *incremental* and feel they can change their intelligence or achievement depending on factors over which they have some control. Students with incremental beliefs even appear to sacrifice opportunities to look smart in favour of learning something new.

There are direct implications of this for the teaching of low-achieving students. Those students who adopt the entity theory of intelligence could do so as a direct result of significant adults, such as parents and teachers, who exaggerate the positives and hide negative information from them. Dweck claimed that, by their actions, some teachers teach students that they are entitled to a life of easy, low effort successes, but she argued that this is a recipe for anger, bitterness and self-doubt. Dweck suggested that some teachers respond to students experiencing difficulty by providing easier tasks for them, thus reinforcing low achievement. These ideas are similar to what Brophy (1983) called a 'self-fulfilling prophecy', as was described in Section 7.

On the other hand, teachers and parents who seek to foster incremental views of intelligence would try to help children to overcome their deficiencies. Dweck suggested that children should know they have our respect, but that self-esteem is not something that can be easily given to them. She argued that it is possible for both praise and criticism to be positive and negative. For example, Dweck suggested that teachers should endeavour to create the impression that deficiencies are a sign to students that they need to try harder, and be frank with students about what they lack and what they need to reach their goals. Students should be encouraged to learn that challenge and effort enhance self-esteem and are not threats.

There is no known direct relationship between the type of goals that students have and their achievement. But high-achieving students tend to have a reserve of successes on which they draw for their confidence and motivation, while low-achieving students experience higher risk of failure in many tasks and so seek to reduce this risk, by minimising the effort invested in tasks, which is termed 'performance avoidance' (Elliot, 1999).

The value of active teaching for low-achieving students

This subsection suggests that, whether in the context of whole class approaches or individual instruction, the research indicates that active teaching, which is used here to include explicit teaching and direct instruction, results in higher outcome gains with low-achieving students.

There are low-achieving students in most mathematics classes. Gervasoni (2004), for example, found that by the end of their first year at school, some 40 per cent of students are falling behind their peers in at least one aspect of number learning. The number and combinations of domains in which children are 'behind' is diverse. This fact further highlights the complexity involved in assisting them. Gervasoni (2004) proposed that low-achieving students can lose confidence in their ability and develop poor attitudes to learning and to school. She argues that it is this loss of confidence that results in the increase in the knowledge gap between these students and others, and argues that the typical learning experiences provided by the classroom teachers for the class do not enable each student to participate fully and benefit. Ginsburg (1997) also argued that:

> *… as mathematics becomes more complex, children with mathematics learning difficulties experience increasing amounts of failure, become increasingly confused, and lose whatever interest and motivation they started out with.*

(Ginsburg, 1997, p. 26)

It seems that the sooner steps are taken to address the needs of low-achieving students the better.

Rosenshine (1986) used the term 'explicit teaching' to include clear explanations and guided practice. More recently the term is taken to mean more. In the teaching of reading, for example, the term 'explicit teaching' includes: knowing the learner; responding to the learner; implementing focused lessons; and reflection and review (see Edwards-Groves, 2002). Similarly, Hattie (2009), in a comprehensive meta-analysis of results of experimental research, discussed in some detail the positive effects of direct instruction, which he described as involving the teacher having a clear idea of learning intentions, explicit criteria for success, building commitment and engagement with the task, modelling with checks for understanding, guided practice and explicit closure.

A substantial review of approaches to mathematics teaching, reported in the United States of America by the National Mathematics Advisory Panel (NMAP) (2008), formed similar conclusions. The panel was well funded and consisted of leading educators, teachers and psychologists, and substantial support staff. They adopted a rigorous approach to evidence and sought submissions from a breadth of research domains over a 20-month period. They identified and reviewed 26 studies that used randomised control designs to examine factors which supported the achievement of learning-disabled and low-achieving students. Their main conclusion was that active teaching (the term they used was 'explicit systematic instruction') improved performance in computation and solving problems. The panel defined 'active teaching' as involving the teacher explaining and demonstrating specific strategies, allowing students many opportunities to ask and answer questions, and encouraging students to think aloud about the decisions they make while solving problems. The NMAP also proposed that teachers recognise the importance of sequencing problems carefully and giving clear feedback to students on the accuracy of their work.

These findings on active teaching are compatible with findings from other research syntheses of pedagogical approaches. Hattie and Timperley (2007), for example, analysed responses from a wide range of studies addressing the needs of low-achieving students and found that interactive teaching along with feedback to students on their learning produced significant positive effects. Hattie and Timperley explained that feedback involves making explicit to students where they are going, how they are going, and where they are going to next.

Similar conclusions were found in *Australian Educational Review No. 48* focusing on instruction for low-achieving students. Ellis (2005), drawing on the psychological literature on teaching students with learning difficulties, argued that teaching of students experiencing difficulty should emphasise explanations, with scripted presentations that include rapid pacing and drill. She argued that such explicit instruction is significantly more effective for teaching low-achieving students than what she termed 'constructivist instruction', which is taken to imply that the teaching and learning are more interactive than directed.

A simplistic reading of the findings in this subsection might result in misinterpretation of the implications. These findings should not be taken as calls for a return to drill-orientated approaches, with the teacher doing most of the talking. On the contrary, the findings are supportive of greater student active engagement in their own learning. For example, the finding from the NMAP suggesting that the teacher explain and demonstrate specific strategies should not be interpreted as the teacher telling students how to perform tasks and procedures, but that the teacher plays an active role before, during and after the learners' goal-directed activity. Learner activity is intended to focus on tasks with specific mathematical intent that has been explained to the students. Teachers are advised to have specific mathematics goals for all of the activities they do with students. Likewise, implementing the NMAP recommendation that teachers allow students many opportunities to ask and answer questions is best done in a supportive classroom community, while students are working on suitable tasks. Further, the suggestion from NMAP that teachers encourage students to think aloud about the decisions they make while solving problems should be read as meaning that low-achieving students are working on solving problems for themselves and not merely performing actions as directed by the teacher.

There are instances of structured programs that are based on the active teaching approach. An example of such an approach from the United States of America, *MathWings* (Madden, Slavin & Simons, 1997), places emphasis on the importance of inclusive classroom teaching, with the aim of improving the experience of all learners so that students experience success in the mainstream. Another Australian program based on active teaching both in classrooms and in withdrawal intensive settings, termed *QuickSmart*, was reported by Graham, Bellert, Thomas and Pegg (2007). Their findings showed impressive improvements in literacy and numeracy learning were experienced by students across a range of levels and settings. Graham et al. described a four-phase process for addressing the needs of low-achieving students involving initial teaching, subsequent attempts to address difficulties experienced by some students, collaborative support for teaching by a specialist and, ultimately, withdrawal from class. The fourth phase of this process is further discussed in the following subsection.

Taken together these findings and recommendations suggest that carefully designed active teaching is likely to maximise learning opportunities for low-achieving students when working within their usual classes. Such teaching also has the effect of communicating to low-achieving students that teachers expect them to learn, which is connected to the discussion of self-fulfilling prophecy and self-efficacy that were discussed in Section 7.

Small group and individual support for low-achieving students

From a student confidence and a school resource perspective, regardless of whether or not classes are heterogeneously grouped, it is best if low-achieving students are taught using whole class approaches that accommodate their learning needs using the approaches similar to those described in Section7. However, when students fall so far behind that they are no longer able to learn within the group, some intervention involving individual or small group attention is needed. In this small group the goal is to accelerate the student's learning to a point where they can better participate and benefit from whole class teaching. Such interventions may or may not involve withdrawal from class, and this may be dependent on the resources available at the school. It should be noted that approaches in which students are withdrawn for short periods are not intended to isolate the student from their class group, but to prepare them for more effective participation in their regular class. It should also be noted that such approaches have the advantage of reducing the possible negative effects of achievement grouping, as was discussed in Section 7.

A well-established research-based program, called *Mathematics Recovery*, which was developed in Australia, engages low-achieving children in the second year of schooling in long-term individualised teaching with the aim of advancing the students' arithmetical learning. These low-achieving students take part in an intensive, individualised teaching program aimed to advance them to an average level. This program has produced impressive results for students, both in Australia (Wright et al., 2000) and in the United States of America (Cobb, 2005).

Another Australian program, the *Extending Mathematical Understanding* program (EMU) (Gervasoni, 2004), is an intervention program for six- and seven-year-old children who are at risk in aspects of number learning. Specialist tutors work with groups of three students in withdrawal settings, but within a whole school approach, as there is a clear intention that the withdrawn students will return to the main classroom. This intervention process is situated, in which schools work on developing whole class strategies so any withdrawal is implemented with the clear intention that students will return to the main classroom. The withdrawal approach is only implemented when some students are so far behind their peers that it is unreasonable to expect the teacher to accommodate their learning needs as well as those with no learning difficulties, within the one class.

In 2000, the effectiveness of both small group and individual *EMU* intervention program structures were trialled, with small groups found to be more effective. In her evaluation of the program, Gervasoni (2004) argued that the *EMU* intervention program provided students with a different level of interaction with the teacher than is possible within the classroom setting during mathematics lessons. Observations of more than 30 EMU sessions in 2000 demonstrated that within each 30-minute session, students and teachers engaged in more than 100 interactions focused on the mathematical ideas investigated during a session. The improvement in students' assessment results and their subsequent class participation indicates the success of this program.

QuickSmart (Graham et al., 2007) is another example of a structured initiative that recognises that students who have fallen behind cannot overcome their weaknesses without explicit support. The withdrawal aspect of the *QuickSmart* program involves students who have performed in the lowest bands in screening tests and who have experienced persistent difficulty in numeracy in structured sessions outside normal classes for three 30-minute sessions for 30 weeks. The intention is that this is supplementary to the usual mathematics classroom, and the withdrawn student continues to participate in the normal classes as well as the tutoring session elements. Analyses of the learning gains of particular cohorts on skill and problem solving based tests indicated that the gains of tutored students were well above those of equivalent non-tutored students. The improvement in performance of Indigenous students after participation in this withdrawal program was particularly impressive, and was also associated with subsequent improvement in attendance and engagement in schooling generally.

Another small withdrawal group initiative, where it was again intended that students would return to their class, but having a target group different from that of *QuickSmart* programs, was reported by Breed and Virgona (2006). These researchers worked in Victoria with small groups of low-achieving junior secondary students for one hour per week for 18 weeks, building on a defined framework of multiplicative learning. Their view was that multiplicative thinking is the basis of much secondary school mathematics and students' misconceptions are a significant barrier to their full participation in mathematics. The data they presented included written assessments, interviews, drawings and ranking of factors, and they reported that students from the intervention group had overcome the deficits in their achievement relative to the class, whereas other untutored but at-risk students did not improve. It also appeared the intervention students showed no signs of the self-fulfilling deficit approach to their mathematical studies, sometimes associated with identification as low-achieving students.

An intervention that has a quite different premise and takes a unique approach, termed *Getting Ready in Numeracy*, was implemented in 2010 in the Western Metropolitan Region in Victoria. It allocates a tutor to small groups of students to prepare them for the mathematics they are expected to learn *before* the lesson in which the whole class will be taught it. The students are withdrawn from a non-mathematics subject and tutored away from the classroom. The purpose of the tutoring is to get the students ready for the mainstream class. To achieve this objective, the particular foci of the tutoring sessions are the formal terminology and other language that will be used in the class, and the format and mode of representing the mathematics that will be used. This has the effect of reducing the cognitive load (Sweller, 1994) of the tutored students when they participate in the whole class instruction, thus increasing the chances that they can interpret the content when it is presented in the class. Some initial familiarity with the content increases the confidence of the tutored students to participate actively.

The researchers found that a benefit of this withdrawal program was that, as there are fewer disruptions from the tutored students subsequent to the tutoring, so the whole class lessons progressed better – thus achieving an equity benefit for all students. Sullivan and Gunningham (2011) present a range of qualitative data, including very positive responses from teachers, tutors and students, and quantitative data which indicate strong learning gains by the tutored students.

While all of these individual and small group approaches are resource intensive, the reports cited here present evidence of their success and it seems that the identification of the necessary resources at an early stage does produce a positive return on investment later, for the individuals, groups and eventually for society.

Particular learning needs of Indigenous students

In the results of the international assessments described in Section 2, data demonstrated groups of students who performed substantially below their peers, with the largest difference being between Indigenous and non-Indigenous Australians. There is substantial community interest in addressing disparities between Indigenous and non-Indigenous Australians on a range of social indicators, and education is, arguably, the most appropriate focus for seeking to address systematic inequities. In the most recent 2009 PISA survey, on average, Indigenous students were around two years (76 points) behind non-Indigenous students. Given the links between mathematics achievement and employment and study options, this represents a significant structural disadvantage experienced by many Indigenous students. This subsection describes some approaches to addressing this structural disadvantage, not only to suggest approaches that may be effective for teachers in schools with high proportions of Indigenous students, but also because those same approaches will be useful in assisting other categories of disadvantaged students. Note should be taken that there are high-achieving Indigenous students and also that there are substantial numbers of non-Indigenous students who are experiencing difficulty learning mathematics. Seeking insights into successful initiative with Indigenous students may have potential to indicate possible avenues for solutions for these other cohorts as well.

Jorgensen and Sullivan (2010) described some factors which contribute to the general educational challenges faced by Indigenous students, including irregular attendance of some students, cultural aspects of language usage, specific aspects of the technical language used in mathematics classes, ways of interpreting time and space, and for some Indigenous students the impact of living and learning remote locations. Mellor and Corrigan (2004) outlined the impact of this cluster of factors, and the effectiveness of policy since then to reduce the powerful grip they have on the learning outcomes of most Indigenous students appears to be slight.

There are two types of projects described: those that use specific pedagogical adaptations to accommodate the cultural background of Indigenous students; and those that make recommendations about practices in Indigenous schools, but which are more generally applicable.

Culturally sensitive approaches improving Indigenous mathematics education

One stream in the literature on ways of improving mathematics teaching for Indigenous (and other cohorts of low-achieving) students involves identifying particular characteristics of the culture of the students and using those characteristics to inform pedagogy. Because most initiatives reported in the following are descriptions of the projects rather than analyses of the impact on student learning, the projects are each described briefly with the intent of seeking common themes.

Howard (1997) and Cooper, Baturo and Warren (2005) argued that a conventional curriculum, such as those used in Australian states and territories, can be alienating for many

Aboriginal children. The *Garma* Mathematics Curriculum (2007) exemplifies the value of making mathematics more accessible, connected, and meaningful to Northern Territory Indigenous students. The *Garma* program is conceptualised as two-way learning (a mixing of modern and Indigenous knowledge) and incorporates aspects of the Yolŋu kinship system, makes connections between an Indigenous sense of place and concepts of pattern and space and conventional representations of this aspect of the curriculum.

A second approach to addressing the cluster of factors which play out with Indigenous learners is one which recognises that the nature and quality of parental support with homework and developing positive attitudes to schooling that many non-Indigenous students experience is not always available for Indigenous students. An explicit focus of these initiatives has been to build connections between the school and the relevant communities.

Engagement with the relevant communities was a strategy that was specifically adopted in the *Mathematics in Indigenous Contexts* program, with participating (K–2) students in project schools improving on pre-test scores on the New South Wales Schedule for Early Number Assessment, as evaluated by Erebus International, an Australian commercial evaluation and management development practice, in 2007. The *Make It Count* project (Morris & Matthews, 2011) currently being conducted under the auspices of the Australian Association of Mathematics Teachers, has as its explicit focus the development of relationships of the school with Indigenous communities, and is seeking ways to incorporate community perspectives into decisions about curriculum and pedagogy.

Promoting such community engagement initiatives should be expanded for Indigenous communities, but this general strategy is also helpful in enhancing the learning of other groups of disadvantaged students. However, some of the challenges associated with engaging Indigenous communities with their local schools are somewhat different from those with other communities. But schools which serve any disadvantaged group need to recognise the alienation from school commonly experienced by members of such communities and take specific actions, appropriate to those groups to seek to engage parents and others in school decision making.

There have also been studies that have argued that Indigenous ways of knowing can assist in the learning of modern mathematics concepts. Robert Reeve (2010), at the *Teaching Mathematics? Make it count* conference reported that the visual memory skills demonstrated by many of the Indigenous students studied could be used in supporting student recognition and learning of number. Reeve tested speakers of Walpiri and Anindilyakwa, in two remote sites in the Northern Territory, and subsequently argued that the teaching of number that relies solely on fluency with number words, that is relying on literacy rather than visual skills, make the learning more inaccessible for some Indigenous students. His presentation demonstrated that including a variety of representations in tasks, especially visual, helps with learning.

Together, these projects argue for specific consideration of the needs of Indigenous learners and the adaptation of curriculum and pedagogies to accommodate those needs. It also seems that initiatives to connect schools with Indigenous communities have been successful, at least in bringing the communities to the schools.

Generally applicable pedagogic approaches to teaching Indigenous students

The second stream in the research is from projects that seek to identify organisational and pedagogical strategies that seem to be working well in Indigenous contexts, but which are clearly applicable more generally. Frigo, Corrigan, Adams, Hughes, Stephens and Woods (2003) reported on a study, conducted under the auspices of the Australian Council for Educational Research, of schools with high proportions of Indigenous students, which included analysis of the Indigenous students' achievement in literacy and numeracy over the years 2000–02. In particular, positive outcomes were associated with strong school leadership, in partnership with local Indigenous leaders, the presence of adults from the Indigenous community in the school and specific actions to support regular attendance and the active engagement of students in their learning. Frigo et al. (2003) listed key elements of effective numeracy teaching from across schools with high proportions of Indigenous students as being to:

- teach skills in real-life contexts
- develop sound number skills
- reinforce concepts through structured activities and semi-structured play
- offer low-risk opportunities to develop confidence
- explore the language of mathematics
- build on what the students already know.

Similarly, Erebus International (2007), which evaluated a range of Indigenous programs recommended that teachers recognise the individuality of students, provide a rich language environment, contextualise learning activities, identify what individuals know and need to know, and develop positive relationships with students.

The *Maths in the Kimberleys* is a multi-stranded, classroom-based, research project, aiming for a community-coherent approach, using focused teaching and sensitive pedagogy (Grootenboer, 2009; Jorgensen & Sullivan, 2010; Sullivan, Youdale & Jorgensen, 2010). One strand of this research involves the use of collaborative group-oriented pedagogies in which students work together on rich tasks, with a socially coherent emphasis on mutual responsibilities, use of home language, and reporting to the class as a learning opportunity. (For an elaboration of a similar approach in the context of a multilingual school in the United States of America, see Staples, 2008).

A second strand of this project, building on assessments of the students' prior learning, examined student achievement following the focused teaching approaches that involved active engaging experiences that incorporate many of the six principles of mathematics teaching described in Section 5. In particular, Sullivan et al. (2010) identified factors contributing to student learning gains, including when teachers articulated the goals of teaching to the students and used thoughtfully sequenced activities that built on students' language and experience. Students in the observed classes of project teachers were both willing and able to engage with rich tasks that required decision making and allowed the construction of mathematical ideas by students.

The projects mentioned in this subsection all make recommendations about pedagogies that have been shown to improve learning in Indigenous context, that also happen to be applicable to all students.

Concluding comments

This section has argued that it is essential for teachers be aware of, and plan for, differences in mathematics achievement of students, and to actively adapt their practice in ways to address issues associated with diversity and equity. It argued that both curriculum and pedagogy should be adapted to accommodate the interests and needs of the particular cultures and backgrounds of the students. Most mathematics classes have low-achieving students and some classes have many such students. Schools and their mathematics teachers need support in developing a range of approaches that work well with intact class groups. Improving the learning opportunities of all students, even though the focus is on low achievers, is the goal of all the approaches described in this section.

The positive impact that teachers and parents can have on student motivation was described. While understanding the impact of motivation is important for students, low-achieving students are particularly at risk in so far as their inappropriate motivation may inhibit their learning opportunities.

Active teaching, as described in this section is helpful for low-achieving students, and all teachers should be aware of pedagogies associated with such teaching. It was noted that many mathematics educators see active teaching as connected to the drill-oriented approaches adopted by some tutoring franchises, but the research on active teaching indicates that adopting interactive teaching approaches, with an explicit mathematical focus can assist low-achieving students.

Noting the resource demand of intensive withdrawal programs, such initiatives, often of short duration, are needed for some students. They can productively be part of any overall plan to improve the learning outcomes of low-achieving students, and enable more effective learning of other students.

There are particular challenges in supporting mathematics teachers and learners in Indigenous schools, and some pedagogic approaches to this were outlined in the section. In particular there is a need to consider the particular and unique learning needs of Indigenous students, and well as articulating the pedagogies that work well with all students.

Section 9 incorporates some of these ideas explicitly in the discussion and the recommendations it makes about teacher professional learning.

section

9

Mathematics teacher learning

An important goal of this review is to identify aspects of mathematics teaching that can productively be incorporated into formal and informal teacher learning, so that more of it can be 'made to count'. Previous sections have presented research-based discussion and exemplars of best-practice pedagogy for consideration by prospective and practising teachers. This section will first present a research-based view on the nature and role of knowledge necessary for the effective teaching of mathematics. Second, it will examine school-based approaches to improvement which can be sustained over time and which have been evaluated through research. Third, and building on the previous discussion, the section will describe and propose a structured approach to systemic planning for mathematics teacher learning that draws on many aspects of research and discussions presented earlier in this review paper.

In most cases, the considerations and implications for mathematics teacher education presented in this section apply to both the education of prospective and practising teachers. Even though the majority of the structured learning of prospective teachers is in a university setting and the learning of practising teachers mostly happens in schools, the nature of the knowledge being fostered, and the emphases within that knowledge are similar.

Knowledge for teaching mathematics

It goes without saying that mathematics teacher learning experiences are intended to augment what the teacher already knows about mathematics and its teaching. The categorisation of teacher knowledge proposed by Hill, Ball and Schilling (2008) can assist in detailing a description of this knowledge. Building on the well-known work of Shulman (1987), they proposed two categories of knowledge: subject matter knowledge and pedagogical content knowledge. Hill et al. presented a diagrammatic representation (p. 377) in which pedagogical content knowledge has three sub-categories:

- knowledge of content and teaching
- knowledge of content and students
- knowledge of curriculum.

The meaning of the pedagogical content knowledge sub-categories have been elaborated in various places in this review, especially Sections 5 and 6 (for content and teaching), Sections 3, 7 and 8 (for content and students), and Sections 2 and 4 (for curriculum). Beswick, Callingham and Watson (2011) reported on an Australian-based survey and proposed an interesting hierarchy of aspects of teacher pedagogical content knowledge. Hill et al.'s three aspects of pedagogical content knowledge are essential elements of any professional learning for practising mathematics teachers, and can be assumed to be part of the mathematics education studies of prospective teachers.

The subject matter of mathematics knowledge, according to Hill et al. (p. 377), comprises three sub-categories:

- common content knowledge
- specialised content knowledge
- knowledge at the horizon.

The first two of these sub-categories of mathematical knowledge are contentious and the following discussion seeks to elaborate them. The task involving music cards presented earlier in this review paper (as Figure 6.2), will be used to illustrate the discussion. The task was:

Figure 9.1

> If one pre-paid card for downloading music offers 16 songs for $24, and another offers 12 songs for $20, which is the better buy?

Common content knowledge for mathematics

Basically common content knowledge is the knowledge that mathematically proficient citizens might use in solving problems or interpreting the world. For example, to find a solution to the music cards task, the *common content knowledge* needed might (but not necessarily) involve the application of a known algorithmic procedure. In this case, examples of algorithmic or procedural knowledge required for a solution include cross-multiplying the fractions before comparing, or making one term an unknown and then solving the problem of the form $x/b = c/d$. This type of knowledge is of fundamental importance for mathematics teaching, and effective access to this knowledge is critical for both prospective and practising teachers.

There have been various studies undertaken with prospective primary teachers that have suggested that their common content knowledge is low (Morris, 2001; Hill, Rowan & Ball, 2005) and, while there have been few studies of common content knowledge of practising teachers, it can be assumed theirs is low as well. The challenge for teacher educators in this is that the range of content that would have to be (re)taught to prospective and practising teachers to anticipate all possible practical situations (such as the music cards) is substantial.

Assuming that prospective teachers, for example, study only one unit in their initial training that focuses on common mathematical content knowledge, a key objective of such a unit should be to develop an orientation in the prospective teachers to identify the strengths and weaknesses in their common content knowledge, and to provide them with strategies for learning the mathematics they will need, when they need it. Such an approach would be a significant change to the emphases in many units currently offered to prospective teachers, which are often not much more than a sequence of introductory mathematics topics.

Indeed, similar approaches are needed for practising teachers. In other words, when planning or delivering professional development to practising teachers, rather than trying to ensure that all practising teachers know the full range of mathematics they may need, it would be useful for teachers to develop the skills and resources to be able to find the common mathematics content they need when they need it.

Specialised content knowledge

The term ‘specialised content knowledge’ refers to the knowledge needed by mathematics teachers, but not necessarily expected of mathematically proficient citizens. Note that the term ‘specialised’ here refers to the content knowledge that is specialised for teachers, whereas in Section 2 the term was used to refer to specialised mathematics.

The specialised content knowledge required for teaching the music cards task includes knowing that intuitive strategies can be used for finding an answer, and that there are many different strategies that can be used in solving such a problem. The knowledge also includes awareness that there is interesting mathematics in the intuitive strategies that might be suggested by students, that listening to and clarifying students' strategies is indeed a key aspect of teaching mathematics and that it is not so much the answer as the approach that should be the focus of learning when using such a task.

The emphasis on intuitive strategies being proposed here also allows the development of important generalisations about approaches to such problems that can be taught and learnt. For example, ideally prospective and practising teachers would become aware that the music cards task illustrates the ways unit comparisons are a standard way of approaching any 'best buy' type problems, and that in this case, as in most situations, there are two types of unit comparisons, as were elaborated in Section 6. It is in the awareness and use of diverse strategies that specialised content knowledge for teachers differs from common content knowledge.

For teaching, teachers need to know the principles underpinning various approaches whatever the context and whatever the numbers involved. This knowledge will be useful in every class the teachers teach, and therefore it is most critical they have such understandings, and these understandings should be the basis of learning to teach mathematics, along with how such knowledge informs planning and teaching. These aspects can be addressed in both the formal mathematics studies and the mathematics education studies undertaken by prospective teachers, and should also be emphasised in programs for practising teachers.

Approaches to teacher development that sustain teacher learning

One of the ongoing themes in the mathematics teacher education literature is the difficulty of finding ways to foster and sustain teacher improvement. Successful sustaining strategies are those in which teachers continue to participate, even in the absence of external incentives, and which become part of ongoing, collaborative, school-based, teacher professional learning, involving the study of pedagogical practice. Some approaches that have received widespread recognition include the study of dilemmas that problematise aspects of teaching; and *Learning Study* that engages groups of teachers, both prospective and practising, in thinking about student learning through studying specific examples of practice (Runnesson, 2008).

Similarly, an approach commonly used for collaborative teacher learning in Japan involves teachers thinking together about their long-term goals for students, developing shared teaching–learning plans, encountering tasks that are intended for the students, and finally observing a lesson and jointly discussing and reflecting on it. This approach has also been successfully adapted for the United States of America (Lewis, Perry & Hurd, 2004).

A simplified description of the approach, based on Inoue (2010, p. 6), is as follows:

- a group of teachers plans a lesson together
- one person teaches, the others watch and write reviews
- the lesson plan is revised after group discussion
- a different teacher teaches, others watch and write reviews.

This process cycles through, over and over. Of course, a major challenge in this for Australian teachers is having a second teacher observing their teaching, since there is a strong culture of privacy associated with classroom teaching in this country. Nevertheless, if this barrier can be overcome, by building trust between teachers and emphasising an orientation to improvement as distinct from evaluation, this approach will result in powerful mathematics teacher learning. Collaborative approaches, where the focus is on improvement of lessons, as distinct from judgements about teachers, is likely to have longer terms benefits for groups of teachers.

Such structured collaborative approaches are more difficult in the pre-service settings than in schools since the requirement for long cycles of review and reflection is more difficult to achieve. Nevertheless, the principles of collaborative planning, with observation and review of the lesson rather than the teacher, can be effectively incorporated into the practicum experiences of prospective teachers.

Considering systematic planning for teacher learning

One of the criticisms levelled at many initiatives for mathematics teacher learning is that there does not seem to be readily identifiable principles that guide the design and emphases of such programs. This subsection will synthesise the themes presented in this review and describe them in terms of teacher learning.

An important set of principles that can be used to guide the design and delivery of mathematics teacher professional development for practising rather than prospective teachers was proposed by Clarke (1994, p. 38) and can be summarised as follows:

- address issues of concern and interest to the teachers
- involve groups of teachers from a school including the school leadership
- recognise impediments to teachers' growth
- model desired classroom approaches during in-service sessions
- enlist teachers' commitment to participate
- that changes are derived largely from classroom practice
- teachers should be allowed time to plan and reflect
- engage teachers as partners
- recognise that change is gradual.

Of course, most if not all of these points apply to teacher professional learning generally.

Each of these characteristics of professional learning programs is important, especially in the priority given to enlisting the commitment of teachers to the professional learning, the collaborative nature of such learning and connections with practice, and each can be productively acknowledged in planning mathematics teaching learning. In other words, teacher professional development that is imposed and externally designed is less likely to improve teaching practice than initiatives in which the teachers are involved in all aspects of design, delivery and evaluation.

A key step is presented in the second of Clarke's (1994) principles of professional development; that of involving and enlisting the support of all levels of school leadership. Noting that teacher improvement is difficult (see the special issue of *Journal of Mathematics Teacher Education* edited by Brown, 2010), school and faculty leadership involvement in professional development is critical and their commitment to the principles of the teacher learning program must be explicit. These observations are complementary to those expressed by Mulford (2005) and much of the educational leadership research. Connected to this point are two other aspects that seem to be central:

- whole learning team involvement is essential since change does not mean tinkering at the edges, but examining all aspects of planning, teaching and assessment
- support must be provided in terms of allocating time to engage with the program and time to implement suggested ideas, as well as the provision of the necessary resources.

This review paper proposes that there are four complementary, but different, strategies or emphases in teacher professional development. They are described and analysed separately in the following text.

Strategy 1: Creating possibilities for engaging students in learning mathematics

There have been many issues identified and analysed in this review paper that are important for all teachers of mathematics, and which can form the basis of structured professional learning. They are:

- examining the development of the 'big ideas' that underpin the main strands of the mathematics curriculum, and being able to use the content descriptions of new Australian mathematics curriculum to inform long-term and daily planning
- exploring the meaning of the mathematical actions of Kilpatrick et al. (2001) which are presented as proficiencies in *The Shape of the Australian Curriculum: Mathematics* ACARA (2010a) and devising experiences for students that create the possibility of all four proficiencies: understanding, fluency, problem solving and reasoning
- ways of appropriately emphasising numeracy and practical mathematics in teaching and assessment in the compulsory years
- approaches to engaging all students through increasing opportunities for decision making, connecting learning to their experience, and illustrating the usefulness of the learning
- selecting and using a range of tasks that engage students in meaningful mathematics and numeracy and building these tasks into lessons
- exploring the specialised content knowledge involved in mathematical tasks, and developing strategies for identifying aspects of common content knowledge that may be needed, including strategies for learning that knowledge when it is required
- examining pedagogies that are appropriate with heterogeneous classes, including specific actions to support students experiencing difficulty and to extend those who are ready.

These elements are central to the mathematics education components of the education of prospective primary and secondary teachers Due to the scope of this content, teachers will require substantial formal input through lectures and readings, and classroom-based reflection and practice. These elements are also the basis of structured professional learning for practising teachers. For such learning to occur, formal input over time will be required, opportunities to engage with ideas in simulated situations, trialling in classrooms and reporting back to peers.

Strategy 2: Fostering school-based leadership of mathematics and numeracy teaching

There were actions identified in this review paper that are best developed within school-based teams, and for which the mathematics teachers require significant school-based leadership. Those actions include issues associated with task use, the development of lesson structures, and the six principles for effective mathematics teaching. It is proposed that there should be particular programs offered for current and prospective leaders of teachers of mathematics, both those working in schools and those working in coaching or consultancy roles. These professional development programs could include the following strategies.

- examining processes for supporting teacher professionalism, the building of relationships, and the development of a learning culture
- collaborative and sustained teacher learning through review of practice
- appreciating the role of evidence in evaluating and supporting teaching and learning, including approaches to assessment and reporting
- the six principles of teaching that can serve to prompt teacher learning
- encouraging teachers to undertake professional reading and ways of identifying suitable sources of such reading
- inducting teachers of other subjects into the principles and processes of numeracy across the curriculum.

Of course, school-based mathematics leaders need ongoing support, including making time available for meetings, to enact some of these actions with teacher planning teams. And such programs also need to be funded and implemented in an ongoing manner so that the school and teachers can plan to maximise the benefits offered to their students and teachers by such programs.

Strategy 3: Choosing and implementing an appropriate intervention strategy

As indicated in Section 7, it is unreasonable to expect classroom teachers to address the needs of learners who have fallen many years behind the expectations for their class. Many schools need to have a strategy for supporting such students to reduce the gap between them and their peers. An effective intervention strategy requires:

- a clear rationale for the program, including ways of identifying target students
- structured learning for tutors or anyone supporting such students, especially on strategies that can engage reluctant learners in small group situations, effective ways of communicating and modelling mathematics, group size, intervention frequency, duration
- specific learning for teachers of the students who are being tutored
- commitment from school leadership and understanding of the nature of the program.

Schools that enact such intervention programs must recognise that there are time and resource implications in such initiatives. Two further actions which schools should commit to and undertake consequent to implementing any intervention, are to ensure there are systematic ways of monitoring the learning of the students who are being supported in this way, and an ongoing commitment to supporting the tutors and teachers who are involved in the program.

Strategy 4: Out-of-field teachers

Given that the number of mathematics graduates applying for teaching positions is less than is needed to cover all mathematics classes, it can be anticipated that there will be teachers of mathematics in schools whose main interest and education is in another domain. These teachers need access to the learning experiences that are listed in Strategy 1. They will also need access to particular support on aspects of mathematics that may be unfamiliar to them, and the orientation to learning the mathematics appropriate to specific specialised knowledge they lack. They will also need assistance developing pedagogies that allow both teachers and students to explore mathematical ideas, to be undertaken in a culture where there is less expectation that the teacher is the one who knows everything.

Together these four components address particular needs of mathematics teachers, and would ideally be offered by systems, regions and networks in parallel.

Concluding comments

This section has suggested that the distinction between common and specialised knowledge helps to delineate priorities and emphases in the mathematics focused education of prospective and practising teachers. It argued that, rather than seeking to reteach all aspects of mathematics, the focus of teacher learning should be on an orientation to and strategies for learning the required mathematics at the time it is needed. It suggested that collaborative teacher learning is a powerful tool that can be used for ongoing and sustained improvement of teaching in schools. It also suggested that there needs to be systematic planning of mathematics professional learning of teachers and proposed four particular emphases in program types, each of which address specific audiences.

section 10

Conclusion

Having sufficient professionals and citizens skilled at using mathematics is directly connected to future national productivity and so it is critical that Australia progressively monitors the mathematics learning of students. Since mathematics creates study and employment opportunities for individuals, it is fundamental that all Australian students should have equitable access to those opportunities.

A range of research indicators presented in this review paper suggest that there is potential for improvement in the learning of mathematics at all levels of the education system. The groups who appear to be in need of specific attention are those disadvantaged by socioeconomic, cultural, gender or geographic factors, or by a combination of several of these factors.

The pathway to improvement is through teacher learning and the most likely format for successful teacher learning is in school-based collaborative teams. Any government actions that may lead to an inhibition of the fostering of a collaborative culture among teachers should be avoided.

This review paper has presented a range of options and possibilities for teaching mathematics. It drew on various sources to describe both what mathematics teaching in Australia seems to be and what it could be. It summarised major recommendations about adapting curriculum and pedagogy, with a view to making the study of mathematics more enjoyable for students, and thus creating an increase in the proportion of school leavers who are successful at mathematics.

The review was informed by contributions at the *Teaching Mathematics? Make it count* conference. It differentiated two main perspectives on the purpose of mathematics teaching: one that draws on practical uses of mathematics, and another that emphasises a specialised interpretation of mathematics. This review paper presented research which addressed these two distinctions and argued that the emphasis in curriculum in the compulsory years of schooling should be on practical mathematics. It also outlined particular mathematical actions that together represent the processes in which students should engage when learning mathematics, noting that the full range of desired actions do not seem to be currently implemented in Australian mathematics classrooms.

Drawing on data from international and national assessments, in Section 3 the mathematical achievement levels of Australian students overall were described. While many students are doing well, there are particular groups of students who are underperforming in comparison with their peers nationally and internationally. The decline in the proportion of students selecting advanced mathematics options at the end of secondary school was noted, although it was suggested that there are still many students completing school mathematics studies and so the reasons for the low enrolments in mathematics studies at university requires investigation.

Since there is substantial community interest in numeracy, related to the economic and social needs associated with mathematics, this review paper has presented a perspective on numeracy that illustrates that, despite what seems to be the common usage, the term 'numeracy' refers to more than a subset of mathematics. It has also argued this broader perspective has the potential to enrich not only mathematics curricula but that it also provides cross-curricular benefits.

Synthesising key ideas from similar lists that have resulted from research, six principles for teaching mathematics were presented in Section 5 that can be summarised as referencing the importance of the teacher having clear goals, building on student readiness, engaging students, presenting a variety of tasks, utilising a lesson structure that encourages students to report on their learning, and encouraging fluency and practice.

In Section 6, the review gave examples of a range of types of mathematics tasks and argued that effective teaching, incorporating a full range of mathematical actions, is dependent on presenting to students important and engaging tasks for which they make their own decisions on solving strategies, rather than following procedures.

Based on the earlier discussion in Section 3 about the diversity of student achievement, an approach to teaching mathematics that includes all students in whole class groups was presented in Section 7, arguing that the negative effects of achievement grouping can be avoided through the adoption of such approaches. The proposition is that the whole class be treated as a community in which all students participate, with the teacher posing variations in task demand for students experiencing difficulty and those who finish the work quickly.

In Section 8 the review paper considered issues associated with student motivation and described approaches for engaging low-achieving students, including active teaching, intervention initiatives and particular programs that support the learning of Indigenous students, all of which addressed student motivation.

Drawing on the various issues addressed in this review paper, specific suggestions were presented in Section 9 that can be used to inform both prospective and practising mathematics teacher education, including programs for all teachers, for leaders of mathematics teachers, for intervention programs, and for out-of-field teachers.

There is an ongoing need for governments to support the professional learning of all teachers of mathematics through structured and systematic programs that are practice focused. There is also an ongoing need for governments to initiate and support research into all aspects of the mathematics education of its future citizens, and it is argued that the elements identified in this review paper provide a useful starting place.

List of 2010 ACER Research Conference papers

At the conference, four keynote, 12 concurrent papers, and six poster sessions were presented (available at http://research.acer.edu.au/research_conference/RC2010/). Downloads of papers presented at the conference, or synopses, are available here.

Keynote papers

Clarke, D. (2010). *Speaking in and about mathematics classrooms internationally: The technical vocabulary of students and teachers.*

Daro, P. (2010). *Standards, what's the difference? A view from inside the development of the Common Core State Standards in the occasionally United States.*

Ernest, P. (2010). *The social outcomes of learning mathematics: Standard, unintended or visionary?*

Stacey, K. (2010). *Mathematics teaching and learning to reach beyond the basics.*

Concurrent papers

Callingham, R. (2010). *Mathematics assessment in primary classrooms: Making it count.*

Dole, S. (2010). *Making connections to the big ideas in mathematics: Promoting proportional reasoning.*

Goos, M. (2010). *Using technology to support effective mathematics teaching and learning. What counts?*

Jorgensen, R. (2010). *Issues of social equity in access and success in mathematics learning for Indigenous students.*

Leigh-Lancaster, D. (2010). *The case of technology in senior secondary mathematics: Curriculum and assessment congruence?*

Lowrie, T. (2010). *Primary students' decoding mathematics tasks: The role of spatial reasoning.*

Mulligan, J. (2010). *Reconceptualising early mathematics learning.*

Pegg, J. (2010). *Promoting the acquisition of higher order skills and understandings in primary and secondary mathematics.*

Reeve, R. A. (2010). *Using mental representations of space when words are unavailable: Studies of enumeration and arithmetic in Indigenous Australia.*

Sullivan, P. (2010). *Learning about selecting classroom tasks and structuring mathematics lessons from students.*

Thomson, S. (2010). *Mathematics learning: What TIMSS and PISA can tell us about what counts for all Australian students.*

Turner, R. (2010). *Identifying cognitive processes important to mathematics learning but often overlooked.*

Posters

Jennings, M. (2010). *First year university students' mathematical understanding.*

Lountain, K., Reinfeld, B., Kimer, P., & McQuade, V. (2010). *Maths for learning inclusion – action research into pedagogical change.*

Morris, C. (2010). *Make it count – Numeracy, mathematics and Indigenous learners.*

Neill, A. (2010). *Processes surpass products: Mapping multiplicative strategies to student ability.*

Waddell, P., Murray, P., & Murray, S. (2010). *Online maths resources – Creating deep mathematical thinking or lazy teachers dispensing 'busy work'?*

White, S., & Szues, D. (2010). *Number line estimation behaviours: Influence of strategy*.

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