Reconceptualising early mathematics learning

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Joanne Mulligan is an Associate Professor of Education and Associate Director of the Centre for Research in Mathematics and Science Education (CRiMSE) at Macquarie University, Sydney. Her background in educational psychology, primary teacher education and mathematics education psychology is combined with early teaching and administrative experience in NSW primary schools. Over the past 25 years her research has focused primarily on the development and assessment of number concepts and processes, word problems, multiplicative reasoning, and pattern and structure with 4- to 9-year-olds. She has made a significant contribution to large-scale Australian government and state-funded numeracy projects since the 1990s (e.g., Count Me In Too; Counting On; the Numeracy Research in NSW Primary Schools' Project; the Early Years Numeracy Research Project (Victoria) and the Mathematical Thinking of Preschoolers in Rural and Regional Australia (DEST). She has also contributed to the development and analysis of numeracy items in the NSW Basic Skills Testing Program and quality assessment tasks for the NSW Quality Teacher Program.

As chief investigator of a current ARC Discovery project, her research aims to reconceptualise traditional views and practices of early mathematical development and learning. Associate Professor Mulligan has developed a range of interview-based assessment instruments based on frameworks of learning that enable in-depth analysis of mathematical growth. Her techniques have potentially significant implications for addressing students' learning difficulties. Current research encompasses a range of projects focused on early mathematical development and professional learning such as the role of technological tools, the use of children's literature, preschoolers' mathematical patterning and mathematics education in Indigenous early childhood contexts. She is also currently leading a NSW DET project, Enhancing Success in Mathematics (ESiM), focused on middle schooling.

Abstract

Over the past decade a suite of studies focused on the early bases of mathematical abstraction and generalisation has indicated that an awareness of mathematical pattern and structure is both critical and salient to mathematical development among young children. Mulligan and colleagues have proposed a new construct, Awareness of Mathematical Pattern and Structure (AMPS), which generalises across mathematical concepts, can be reliably measured, and is correlated with structural development of mathematics.

A current large evaluation study was designed and implemented to measure and describe young children's structural development of mathematics in the first year of schooling, *Reconceptualising Early Mathematics Learning: The Fundamental Role of Pattern and Structure*. An intervention was implemented to evaluate the effectiveness of the Pattern and Structure Mathematical Awareness Program (PASMAP) on kindergarten students' mathematical development. Four large schools (two from Sydney and two from Brisbane), 16 teachers and their 316 students participated in the first phase of a two-year longitudinal study. This paper provides an overview of the background studies that informed the development of PASMAP, describes aspects of the assessment and intervention, and provides some preliminary analysis of the impact of PASMAP on students' representations of structural development.

Introduction

One of the most fundamental challenges for mathematics education today is to inspire young children to develop 'mathematical minds' and pursue mathematics learning in earnest. Current research shows that young children are developing complex mathematical knowledge and abstract reasoning much earlier than previously considered. A range of studies prior to school and in early school settings indicate that young children do possess cognitive capacities which, with appropriately designed and implemented learning experiences, can enable forms of reasoning not typically seen in the early grades (e.g., Clarke, Clarke, & Cheeseman, 2006; Papic, Mulligan, & Mitchelmore, 2009; Perry & Dockett, 2008).

On the other hand, finding more effective ways of establishing the root causes of learning difficulties in mathematics is a key concern. The gap between achievers and non-achievers in mathematics begins in early childhood and becomes wider as students grow older, and there is still insufficient research evidence and little consensus about the underlying causes of underachievement. Despite initiatives and reforms in mathematics education many children do not seem to access the deep ideas and key processes that lead to success beyond school.

The Pattern and Structure Project, initiated in 2001, aims to meet this challenge through a different approach to mathematics learning, beginning with very young children, that reaches beyond basic numeracy to one that cultivates mathematical patterns and relationships. Over the past decade, a suite of studies focused on the early bases of mathematical abstraction and generalisation, has found that an awareness of mathematical pattern and structure is both critical and salient to mathematical development among young children. Mulligan and colleagues have proposed a new construct, Awareness of Mathematical Pattern and Structure (AMPS), which generalises across mathematical concepts, can be reliably measured, and is correlated with increasingly developed structural features of mathematics (Mulligan & Mitchelmore, 2009). Finding reliable and consistent methods for describing the growth of children's mathematical structures and relationships, and utilising children's ideas to develop quantitative reasoning at an optimum age, when they are eager to learn, is central to this project.

What is pattern and structure?

A mathematical pattern may be described as any predictable regularity, usually involving numerical, spatial or logical relationships. In early childhood, the patterns children experience include repeating patterns (e.g., ABABAB …), spatial structural patterns (e.g., geometrical shapes), growing patterns (e.g., 2, 4, 6, 8, …), units of measure or transformations. Structure refers to the way in which the various elements are organised and related including spatial structuring (see Mulligan et al., 2003). Structural development can emerge from, or underlie mathematical concepts, procedures and relationships and is based on the integration of complex elements of pattern and structure that lead to the formation of simple generalisations. For example, recognising structural features of equivalence, $4 + 3 = 3 + 4$ may reflect the child's perceived symmetrical structure (see Mulligan & Mitchelmore, 2009).

Background

There is increasing evidence that structural development is crucial to mathematical reasoning and problem-solving among young children. Failure to perceive pattern and structure may also provide an explanation for poor mathematical achievement. Early assessment of, and intervention in mathematics learning, is considered preventative of later learning difficulties (Clements & Sarama, 2009; Wright, 2003). The quality, scope and depth of both the teaching and assessment of early mathematics are now regarded as critical to future success in the subject (Thomson, Rowe, Underwood, & Peck, 2005).

Research on pattern and structure

Research on early mathematics learning has often been restricted to an analysis of children's developmental levels of single concepts such as counting, but has not provided insight into common underlying processes that develop mathematical generalization (Mulligan & Vergnaud, 2006). However, recent initiatives in early childhood mathematics education, for example, the Building Blocks Project (Clements & Sarama, 2009), the Big Maths for Little Kids Project (Ginsburg, Lee & Boyd, 2008) and the Mathematics Education and Neurosciences (MENS) Project provide frameworks to promote 'big ideas' in early mathematics and science education (van Nes & de Lange, 2007).

This trend is reflected in the increasing body of research into young children's structural development of mathematics and early algebraic reasoning. Algebraic thinking is thought to develop from the ability to see and represent patterns and relationships such as equivalence and functional thinking from the early childhood years (Papic, Mulligan, & Mitchelmore, 2009; Warren & Cooper, 2008). Research in number (Hunting, 2003; Mulligan & Vergnaud, 2006; Thomas, Mulligan & Goldin, 2002; van Nes & de Lange, 2007; Young-Loveridge, 2002), patterning and reasoning (Clements & Sarama, 2009; English, 2004), spatial measurement (Outhred & Mitchelmore, 2000; Slovin & Dougherty, 2004), and early algebra (Blanton & Kaput, 2005; Carraher, Schliemann, Brizuela, & Earnest, 2006; Warren & Cooper, 2008), have all shown how progress in students' mathematical understanding depends on a grasp of underlying structure. Significant concentrations of new research with young children focused on data modeling and statistical reasoning also provide an integrated approach to studying structural development (e.g., English, 2010; Lehrer, 2007).

The Pattern and Structure Project

Early studies on the structure of multiplication and division (Mulligan & Mitchelmore, 1997), the number system (Thomas, Mulligan, & Goldin, 2002), and area measurement (Outhred & Mitchelmore, 2000) focused on analysing and describing structural development in studies of 5- to 12-year-olds. Further research on children's representations of mathematics found that a lack of structural awareness impedes mathematical development and relates to poor representational capacity. Low achievers consistently produced poorly organised representations lacking in structure, whereas high achievers used abstract notations with well-developed structures. Essentially, low-achieving students did not focus on structural features when learning mathematics (see Mulligan, 2010).

A suite of studies that followed, the Pattern and Structure Project, indicated that young children who understand the underlying structure of one mathematical concept are also likely to perceive the structure underlying other quantitative concepts, and can learn to abstract and generalise concepts at an early age. The assessment of first graders found their responses to a range of mathematical tasks could be categorised into four stages of structural development – pre-structural, emergent, partial and structural, with a fifth stage, advanced structural, added with the progression of high-achieving students (Mulligan & Mitchelmore, 2009). The student's stage of structural development was highly consistent overall and reflected their level of mathematical understanding.

The Pattern and Structure Mathematics Awareness Program (PASMAP) was then developed to raise students' awareness of pattern and structure through a variety of well-connected pattern-eliciting experiences. Studies have included an extensive, whole-school project across Kindergarten to Year 6; two year-long, design studies in Years 1 and 2; and an intensive, a 15-week empirical evaluation of an individualised program with a small group of kindergarten children (see Mulligan, 2010).

In related studies, Papic found that preschoolers who are provided with opportunities to engage in mathematical experiences that promote emergent generalisation (an intervention program) are capable of abstracting complex patterns before they start formal schooling (Papic, Mulligan, & Mitchelmore, 2009).

These studies indicate that young children can learn complex mathematical concepts very quickly and effectively by focusing on crucial features of mathematical pattern and structure; visual memory, constructing and representing structures independently of models, and the articulation of 'sameness and difference' was central to this process. However, these findings also supported those of earlier studies in that low achievers failed to perceive structure even in simple mathematical forms such as the properties of a square.

Reconceptualising Early Mathematics Learning

This new study was designed to evaluate the effectiveness of PASMAP on students' mathematical development in the first year of formal schooling. A purposive sample of four large primary schools, two in Sydney and two in Brisbane, representing 316 students from a diverse range of socio-economic and cultural contexts, participated in the evaluation throughout the 2009 school year. Two different mathematics programs were implemented: in each school, two kindergarten teachers implemented the PASMAP and two implemented their standard program. The PASMAP framework was embedded into the standard kindergarten mathematics curriculum. A researcher/teacher visited each teacher on a weekly basis and equivalent professional development for both pairs of teachers was provided. Incremental features of the program were introduced by the research team gradually, at approximately the same pace and with equivalent mentoring for each teacher, over three school terms.

All students were pre- and post-tested with *I Can Do Maths* (ICDM) (Doig & de Lemos, 2000); from pre-test data two 'focus' groups of five children in each class were selected from the upper and lower quartiles, respectively. These 160 students were pre- and post- interviewed using a new version of a 20-item *Pattern and Structure Assessment* (PASA). Intervention-based data included observation notes, digital recordings of their learning experiences and a range of work samples. Student profiles of learning aim to (i) describe the 'tracked' developmental pathway(s) of their mathematical concepts and processes, (ii) analyse the quality of the underlying structural characteristics, (iii) describe salient features or relationships built by the student between components or concepts, and (iv) provide evidence of emergent generalisations and reasoning to support these.

The Pattern and Structure Mathematics Awareness Program Intervention

The program is innovative in its conceptual framework and the way learning experiences are scaffolded, where children are encouraged to seek out and represent pattern and structure across different concepts and transfer this awareness to other concepts. It focuses on fundamental processes such as simple and complex repetitions, growing patterns and functions, unitising and multiplicative structure also common to units of measure; spatial structuring, the spatial properties of congruence and similarity, and transformation (see Mulligan, Mitchelmore, English, & Robertson, 2010). Emphasis is also laid on counting through patterns and measures, the structure of operations, equivalence and commutativity.

Discussion

Preliminary analysis indicates that both groups of students made significant progress in mathematics learning outcomes as described by the state syllabus and measured by the ICDM test. It was not expected that significant differences would be found between PASMAP and regular students on pre- and post-tests scores on this standardised measure. However, initial analysis of qualitative data, tracking of the 'focus' students, indicated marked differences between groups in students' level of structural development (AMPS). Students participating in the PASMAP program showed higher levels of AMPS than the regular group, made connections between mathematical ideas and processes, and formed emergent generalisations. Some of the more able students used one aspect of pattern and structure to build new and more complex concepts. Gradually these connections became more like systems of learning that had common structural features. Goldin in his work with Thomas and colleagues refers to these as autonomous powerful systems that become independent over time (Thomas, Mulligan, & Goldin, 2002).

Some exemplars of students' developing structural features are now described. Students used ten frame cards to promote the structure of ten, spatial and counting patterns, grouping and addition combinations. As an assessment task, they were required to draw the frame from memory, describe how they did this and why the frame was used. Figures 1 to 6 show typical examples of ten frames that have been drawn by six individuals at the same point in the learning sequence. Each figure reflects developmental features of students' awareness and use of the structure of the ten-frame: the use of 2-wise or 5-wise patterns (quinary-based structure), the use of co-linearity (row and column structure) and the construction of addition pairs. Figures 1 to 3 show no recognition of the structure of the ten-frame and its facility, although these students were using ten frames regularly; these students had poor AMPS across a range of tasks. Figure 4 shows awareness of the pattern of fives and Figures 5 and 6 strong structural features.

In another task the children had to recall their use of pattern cards depicting the pattern of squares i.e., 1, 2 × 2, 3 × 3, 4 × 4, 5 × 5 square grid cards. This pattern was linked to prior use of simple grid patterns introduced early in the program and the counting patterns of multiples. Figures 7, 8 and 9 show attempts to draw the pattern from memory, but the structure of increasingly larger squares is not generalised and the number of units is counted or added on individually. Figure 9 shows units aligned but extended uni-dimensionally; this is adding a column rather than recognising the multiplicative structure. Figure 10 shows the student's structural development of the pattern of increasingly larger arrays as squares using the alignment of the 'growing squares'. He also explains the numerical sequence as multiplicative.

Implications

One outcome of the project is to validate alternative developmental paths for young children's mathematics learning. Ultimately this research may provide better pathways for those children who may be prone to difficulties in learning mathematics; that is, those who lack AMPS. Tracking, describing and classifying children's models, representations and explanations of their mathematical ideas, and analysing the structural features of this development are fundamentally important. Our studies indicate that consistent methods for analysing students' AMPS are indeed possible and this process provides a rich basis for assessing and scaffolding students' mathematical development. Our goal is a reliable, coherent model for categorising and describing structural development with aligned pedagogical frameworks.

In the forthcoming Australian National Curriculum (ACARA, 2010), Number and Algebra strands are aligned with Problem Solving and Reasoning Proficiencies. 'An algebraic perspective can enrich the teaching of number … and the integration of number and algebra, especially representations of relationships can give more meaning to the study of algebra in the secondary years. This combination incorporates pattern and/or structure and includes functions, sets and logic'. Further, the integration of measurement and geometry, and statistics and probability brings new opportunities to develop a structural approach. The proposed PASMAP will enable professionals to develop and evaluate a new approach with flexibility – one that integrates patterns and structural relationships in mathematics across concepts so that a more holistic outcome is achieved.

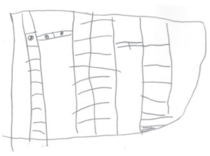

Figure 1: Pre-structural image of 'tall buildings with bridges'.

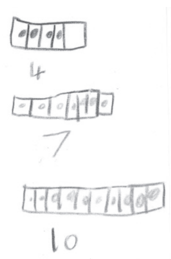

Figure 2: Emergent structural images of single units.

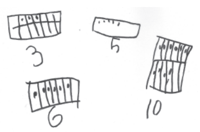

Figure 3: Emergent structural images of 'single and double' frames.

Figure 4: Partial structure shown by 2 × 5 unequal units.

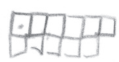

Figure 5: Partial structure: aligned single units ten frame structure.

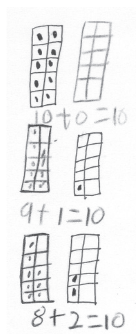

Figure 6: Structural features showing 5-wise pattern.

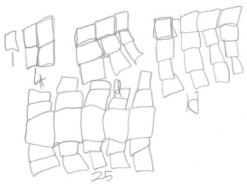

Figure 7: Emergent structure: pattern of squares using single units

Figure 8: Partial structure: pattern of squares using equal-sized units; lack of structure of 'square'

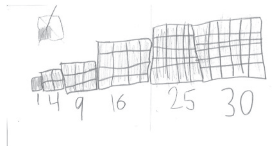

Figure 9: Partial structure: pattern of squares limited to 5x5

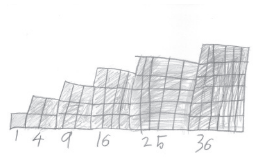

Figure 10: Structural response showing pattern and array structure

Mathematics learning for the future will require young children to reason mathematically in creative and flexible ways in order to solve multi-disciplinary problems. Focusing on pattern and structure may not only lead to improved generalised thinking, but can also create opportunities for developing cognitive capacities commensurate with the abilities of young learners and the demands of mathematics learning for the future.

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