

Identifying cognitive processes important to mathematics learning but often overlooked



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For 13 years prior to that Ross was employed in various roles at the Victorian Board of Studies. He was seconded in 1987 to contribute to redevelopment of the mathematics curriculum and assessment arrangements in the Victorian Certificate of Education. He was appointed to the position of Manager, Mathematics in 1989 and led the implementation of the VCE mathematics study. He was appointed as Manager, Research and Evaluation in 1993. In that role he monitored annual VCE outcomes, and oversaw development and implementation of statistical procedures employed in the processing and reporting of VCE data.

Abstract

This presentation introduces a set of mathematical competencies that deserve to be given more attention in our mathematics classrooms, on the grounds that the possession of these competencies relates strongly to increased levels of mathematical literacy. The presenter argues that widespread under-representation of these competencies among the general populace contributes to unacceptably large measures on the *mathematics terror index*.

The argument in support of these competencies comes out of the OECD's Programme for International Student Assessment (PISA). It is based on the results of research conducted by members of the PISA mathematics expert group. That research will be described, the competencies under discussion will be defined, and the case for greater emphasis on these competencies will be made.

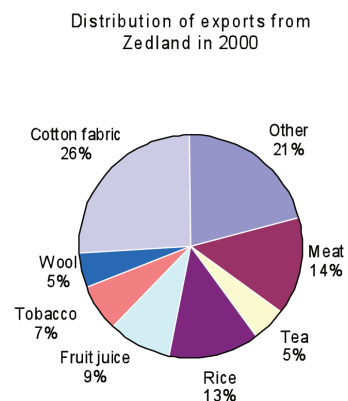
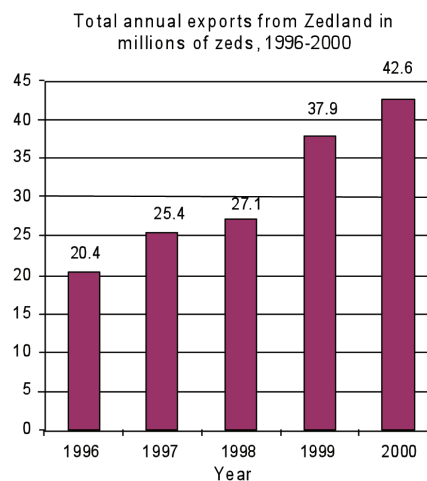
Introduction

The OECD's Programme for International Student Assessment (PISA) aims to measure how effectively 15-year-olds can use their accumulated mathematical knowledge to handle 'real-world challenges'. The measures we derive from this process are referred to as measures of *mathematical literacy*. The *literacy* idea seems to have really taken hold among those countries that participate in PISA. It is generally regarded as very important that people can make productive use of their mathematical knowledge in applied and practical situations.

In this presentation I will demonstrate some illustrative PISA items as a way of introducing a set of mathematical competencies that are fundamental to the possession and development of mathematical literacy, and will propose that these deserve a stronger place in our mathematics classes.

EXPORTS

The graphics below show information about exports from Zedland, a country that uses zeds as its currency.



Q 1: What was the total value (in millions of zeds) of exports from Zedland in 1998?

Q 2: What was the value of fruit juice exported from Zedland in 2000?

A: 1.8 million zeds. B: 2.3 million zeds. C: 2.4 million zeds. D: 3.4 million zeds. E: 3.8 million zeds.

Illustrative PISA items

Two items from the unit titled *Exports* involve interpreting data presented in a bar graph and a pie chart. The first question calls for the direct interpretation of a familiar graph form: identifying that the bar graph contains the required information, locating the bar for 1998 and reading the required number printed above the bar.

The second question is more involved, since it requires linking information from the two graphs presented: applying the same kind of reasoning required in the first question to each of the two graphs to locate the required data, then performing a calculation using the two figures found from the graphs (find 9% of 42.6 million).

A further question *Carpenter* is presented, which requires some geometrical knowledge or reasoning. Familiarity with the properties of basic geometric shapes should be sufficient

Table I: Per cent correct for three illustrative PISA mathematics questions

Question	Per cent correct (all students)	Per cent correct (Aus students)
Exports Q1	67.2	85.8
Exports Q2	45.6	46.3
Carpenter	19.4	23.3

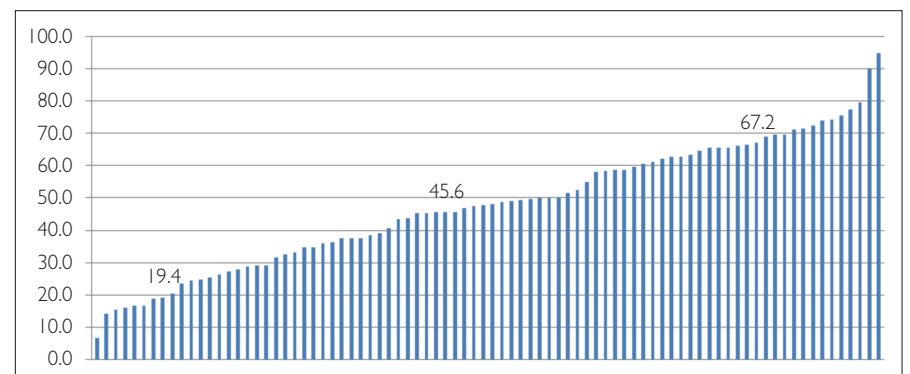


Figure I: International per cent correct of all PISA 2003 mathematics questions

to establish that while the 'horizontal' components of the four shapes are equivalent, the oblique sides of Design B are longer than the sum of the 'vertical' components of each of the other shapes.

What do we find when problems such as these are given to random samples of 15-year-olds across over 60 countries around the world?

Table I presents the per cent correct data for all students internationally and all Australian students who were given the listed questions in the PISA 2003 survey.

The chart in Figure I shows where these publically released questions fit in the context of the whole PISA 2003 survey instrument. The international per cent correct for the illustrative items are labelled, amidst the 84 items used in the survey (with a bar for each item, ordered by their international percent correct value). *Exports Q1* was one of the easier items in the test, while *Exports Q2* was a moderately difficult

CARPENTER

A carpenter has 32 metres of timber and wants to make a border around a garden bed. He is considering the following designs for the garden bed.

A

B

C

D

Circle either "Yes" or "No" for each design to indicate whether the garden bed can be made with 32 metres of timber.

Garden bed design	Using this design, can the garden bed be made with 32 metres of timber?
Design A	Yes / No
Design B	Yes / No
Design C	Yes / No
Design D	Yes / No

item. *Carpenter* was one of the most difficult items.

Is there a problem?

We could speculate about differences in performance levels between Australian and international students, but for my immediate purpose, I might simply suggest that as a mathematics teacher, I would have hoped that most 15-year-olds could answer questions like these correctly. This also has implications for what happens to those 15-year-olds when they leave school, since the mathematical capabilities students demonstrate by the time they are nearing school leaving age foreshadows the approach those individuals will take to using mathematics later in life.

Is the problem that many students don't know the required mathematical concepts; that they have not learned the required mathematical skills? Or could it be that too many 15-year-olds are simply unable to activate the required knowledge when it could be useful; that there is a disconnect between the way in which many of us have been taught, and the opportunities to use mathematics in life outside school?

Usually the opportunities to use mathematics that we come across are not packaged in quite the way they were in school. There, you knew when you were going to a mathematics class. When you went to that class, you did so expecting that you would do things related to mathematics. You had a mathematics teacher who taught and demonstrated mathematical ideas and skills, gave you some examples, and then pointed you to a set of exercises more or less like those used to demonstrate the idea or skill you were learning. You were given instructions like 'count these objects', or 'add these numbers', or 'draw this graph',

or 'factorise these expressions'. The objectives were clearly mathematical.

In the real world, that's not normally how mathematics comes to us. We have to make the judgments and decisions about what mathematical knowledge might be relevant, and how to apply that knowledge. That assumes we are motivated enough in the first place to even notice that mathematics might be relevant.

This brings us back to one of the most important and influential ideas that underpins the PISA project: its emphasis on what is called *literacy*. PISA measures and reports the degree to which the 15-year-olds in participating countries have developed their literacy skills in mathematics and the other survey domains so that they can apply their knowledge to solve contextualised problems – problems that are more like the challenges and opportunities we meet in our work, leisure, and in our life as citizens. But what are the capabilities that equip adults to meet such challenges?

Mathematical competencies – the research

The frameworks that governed the mathematics part of the PISA surveys conducted in 2000, 2003, 2006 and 2009 describe a set of eight mathematical competencies. For the purposes of a research activity we have carried out, these have been configured as a set of six competencies that are fundamental to the concept of mathematical literacy that PISA espouses, namely the capacity to use one's mathematical knowledge to handle challenges that could be amenable to mathematical treatment. Our research has shown that these competencies can be used to explain a very large proportion of the variability in the difficulty of PISA mathematics test items, possibly as much as 70 per cent of that variability. To identify

factors that explain so much of what makes mathematics items difficult is an important finding.

Those competencies can be thought of as a set of individual characteristics or qualities possessed to a greater or lesser extent by individuals. However, we can also think about these competencies from the 'perspective' of a mathematics problem, or a survey question: to what extent does the question call for the activation of each of these competencies? In the following section the six competencies are defined, and the task-level demand for activation of each competency at different levels is described.

Communication

Mathematical literacy in practice involves *communication*. Reading, decoding and interpreting statements, questions, tasks or objects enables the individual to form a mental model of the situation, an important step in understanding, clarifying and formulating a problem. During the solution process, which involves analysing the problem using mathematics, information may need to be further interpreted, and intermediate results summarised and presented. Later on, once a solution has been found, the problem solver may need to present the solution, and perhaps an explanation or justification, to others.

Various factors determine the level and extent of the communication demand of a task. For the receptive aspects of communication, these factors include the length and complexity of the text or other object to be read and interpreted, the familiarity of the ideas or information referred to in the text or object, the extent to which the information required needs to be disentangled from other information, the ordering of information and whether this matches the ordering of the thought processes required to

interpret and use the information, and the extent to which different elements (such as text, graphic elements, graphs, tables, charts) need to be interpreted in relation to each other. For the expressive aspects of communication, the lowest level of complexity is observed in tasks that simply demand provision of a numeric answer. As the requirement for a more extensive expression of a solution is added, for example when a verbal or written explanation or justification of the result is required, the communication demand increases.

Mathematising

Mathematical literacy in practice can involve transforming a problem defined in the real world to a strictly mathematical form (which can include structuring, conceptualising, making assumptions, formulating a model), or interpreting a mathematical solution or a mathematical model in relation to the original problem.

The demand for mathematisation arises in its least complex form when the problem solver needs to interpret and infer directly from a given model; or to translate directly from a situation into mathematics (for example, to structure and conceptualise the situation in a relevant way, to identify and select relevant variables, collect relevant measurements and make diagrams). The mathematisation demand increases with additional requirements to modify or use a given model to capture changed conditions or interpret inferred relationships; to choose a familiar model within limited and clearly articulated constraints; or to create a model for which the required variables, relationships and constraints are explicit and clear. At an even higher level, the mathematisation demand is associated with the need to create or interpret a model in a situation in which many assumptions, variables, relationships and constraints are to be identified or

defined, and to check that the model satisfies the requirements of the task; or to evaluate or compare models.

Representation

This competency can entail selecting, devising, interpreting, translating between, and using a variety of representations to capture a situation, interact with a problem, or to present one's work. The representations referred to include equations, formulas, graphs, tables, diagrams, pictures, textual descriptions and concrete materials.

This mathematical ability is called on at the lowest level with the need to directly handle a given familiar representation, for example translating directly from text to numbers, or reading a value directly from a graph or table. More cognitively demanding representation tasks call for the selection and interpretation of one standard or familiar representation in relation to a situation, and at a higher level of demand still when they require translating between or using two or more different representations together in relation to a situation, including modifying a representation; or when the demand is to devise a representation of a situation. Higher level cognitive demand is marked by the need to understand and use a non-standard representation that requires substantial decoding and interpretation; to devise a representation that captures the key aspects of a complex situation; or to compare or evaluate different representations.

Reasoning and argument

This skill involves logically rooted thought processes that explore and link problem elements in order to make inferences from them, check a justification that is given, or provide a justification of statements.

In tasks of relatively low demand for activation of this ability, the reasoning

required involves simply following direct instructions. At a slightly higher level of demand, items require some reflection to connect different pieces of information in order to make inferences (for example, to link separate components present in the problem, or to use direct reasoning within one aspect of the problem). At a higher level, tasks call for the analysis of information in order to follow or create a multi-step argument or to connect several variables; or to reason from linked information sources. At an even higher level of demand, there is a need to synthesise and evaluate information, to use or create chains of reasoning to justify inferences, or to make generalisations drawing on and combining multiple elements of information in a sustained and directed way.

Devising strategies

Mathematical literacy in practice frequently requires *devising strategies for solving problems mathematically*. This involves a set of critical control processes that guide an individual to effectively recognise, formulate and solve problems. This skill is characterised as selecting or devising a plan or strategy to use mathematics to solve problems arising from a task or context, as well as guiding its implementation.

In tasks with a relatively low demand for this ability, it is often sufficient to take direct actions, where the strategy needed is stated or obvious. At a slightly higher level of demand, there may be a need to decide on a suitable strategy that uses the relevant given information to reach a conclusion. Cognitive demand is further heightened with the need to devise and construct a strategy to transform given information to reach a conclusion. Even more demanding tasks call for the construction of an elaborated strategy to find an exhaustive solution or a

generalised conclusion; or to evaluate or compare different possible strategies.

Using symbolic, formal and technical language and operations

This involves understanding, manipulating, and making use of symbolic expressions within a mathematical context (including arithmetic expressions and operations) governed by mathematical conventions and rules. It also involves understanding and utilising formal constructs based on definitions, rules and formal systems and also using algorithms with these entities. The symbols, rules and systems used will vary according to what particular mathematical content knowledge is needed for a specific task to formulate, solve or interpret the mathematics.

The demand for activation of this ability varies enormously across tasks. In the simplest tasks, no mathematical rules or symbolic expressions need to be activated beyond fundamental arithmetic calculations, operating with small or easily tractable numbers. More demanding tasks may involve direct use of a simple functional relationship, either implicit or explicit (for example, familiar linear relationships); use of formal mathematical symbols (for example, by direct substitution or sustained arithmetic calculations involving fractions and decimals); or an activation and direct use of a formal mathematical definition, convention or

symbolic concept. Increased cognitive demand is characterised by the need for explicit use and manipulation of symbols (for example, by algebraically rearranging a formula), or by activation and use of mathematical rules, definitions, conventions, procedures or formulas using a combination of multiple relationships or symbolic concepts. And a yet higher level of demand is characterised by the need for multi-step application of formal mathematical procedures; working flexibly with functional or involved algebraic relationships; or using both mathematical technique and knowledge to produce results.

The research on these competencies saw a group of experts assign ratings to PISA mathematics items according to the level of each competency demanded for successful completion of each item. Sets of items were rated by several experts, and the ratings were analysed: the average ratings were used as predictors in a regression on the empirical difficulty of the items. The level of demand for activation of these six competencies is an extremely good predictor of the difficulty of the test item.

In Table 2 the competency ratings of the illustrative items presented earlier, assigned by three experts, are reported.

For *Exports Q1*, a relatively easy item, the *communication* and *representation* competencies are the most strongly

demand, with the others demanded little or not at all. The *communication* demand lies in the need to interpret reasonably familiar nevertheless slightly complex stimulus material, and the *representation* demand lies in the need to handle two graphical representations of the data. For *Q2*, the *representation* demand is even higher because of the need to process the two graphs in more detail. Each of the other competencies is also called on to some degree, with the need for *reasoning*, some *strategic thinking*, and calling on some low-level *procedural knowledge* to perform the required calculation.

For *Carpenter*, the *reasoning* required comprises the most significant demand, but each of the other competencies is demanded to some degree.

The message?

Of course this research has further to go; nevertheless, the results of this work are encouraging enough for me to make some conjectures about the importance of this set of competencies, and about how this information might be used in mathematics classrooms:

- Possession of these six competencies is crucial to the activation of one's mathematical knowledge.
- The more an individual possesses these competencies, the more able he or she will be to make effective use of his or her mathematical

Table 2: Competency ratings of three experts for the four illustrative PISA items

Rating (from raters 1/2/3)	Competency					
Item	Communication	Mathematising	Representation	Reasoning and argument	Devising strategies	Symbols and formalism
Exports Q1	1/1/2	1/0/0	1/1/1	0/1/0	0/0/0	0/1/0
Exports Q2	1/1/2	1/0/1	2/2/2	1/1/1	2/0/1	0/1/1
Carpenter	2/2/1	1/0/1	1/1/1	2/3/2	2/1/1	1/1/1

knowledge to solve contextualised problems.

- These competencies should be directly targeted and advanced in our mathematics classes.

In general, not enough time and effort is devoted in the mathematics classroom to fostering the development in our students of these fundamental mathematical competencies. Moreover, the curriculum structures under which mathematics teachers operate do not provide a sufficient impetus and incentive for them to focus on these competencies as crucial outcomes, alongside the development of the mathematical concepts and skills that typically take centre stage.

What actions can be taken to improve this situation?

We must recognise the importance of the fundamental mathematical competencies that I have referred to. These competencies must be given a conscious focus in our mathematics classes, through teaching and learning activities, and through assessment.

In my view, a key place to start is with the nature of discussion that is facilitated in mathematics classrooms. Students need to be given opportunities to articulate their thinking about mathematics tasks and about mathematical concepts. Obviously teachers play a central role in orchestrating that kind of discussion in class and this provides the basis for encouraging students to take the next key step, writing down their mathematical arguments. Giving emphasis to the communication of mathematical ideas and thinking, both in oral and written forms, is essential both to improving communication skills, but also to developing the mathematical ideas communicated and the capacities to use them.