Mathematics teaching and learning to reach beyond the basics

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Abstract

The purpose of this presentation is to paint a broadbrush picture of the challenge of providing mathematics teaching that encourages learning that goes beyond 'the basics'. The presentation focuses on mathematical reasoning and suggests ways in which it can be given a more secure place in Australian mathematics classrooms. Two studies are reported, both of which arose from concern about the 'shallow teaching syndrome' evident in many Australian classrooms where there is very little mathematical reasoning in evidence. One study examined Year 8 textbooks, finding that very few presented 'rules without reasons' and taken overall generally presented a good array of explanations involving reasoning of several distinct types to help students understand why results were true. It was evident, however, that these explanations were generally only used to justify the rule, and were not called upon in any way once it was established. A second study interviewed about 20 leaders in mathematics education to explore their opinions on the shallow teaching syndrome (most – but not all – felt it was a real effect of disturbing prevalence), and the teaching of mathematical reasoning and problem solving. The presentation includes some suggestions for strengthening the place of mathematical reasoning in Australian classrooms and the new Australian curriculum.

Introduction

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Because of their abstractness, learning about the objects with which mathematics is concerned is difficult. Because mathematics is a doing subject, transforming and combining these objects is central, so developing the relevant skills to a high degree of fluency is central. The difficulty of the learning is heightened by the hierarchical nature of mathematics, where skill is built on skill and concept is built on concept. No wonder that learning 'the basics' (the concepts, the skills and how to use them in standard ways to solve problems that relate directly to real-world situations) can easily fill all the time in school devoted to mathematics. Listing the concepts, the skills and their direct applications could also easily fill a whole national curriculum.

Important as the content above is, and despite the tendency for it to appear to define what mathematics is, mathematics is only partially described by such concepts, skills and standard applications. The less visible aspect of mathematics is its process side (how mathematics is done) which for the past nearly 20 years has been labelled 'Working Mathematically' in Australia. In the presentation, I will give a brief overview of the various ways in which this strand has been treated in Australian mathematics in the past, leading up to the current first cycle of the Australian curriculum. Here the elements of Working Mathematically most clearly appear as two of the four proficiency strands: problem solving and reasoning. Neither of these strands seems to be yet operationalised as clearly as will be required if teachers are to be encouraged to pay serious attention to them. This presentation will present ideas on the development of the reasoning strand.

Reasoning in mathematics is a cognitive process of looking for reasons and looking for conclusions. To learn mathematics, students need to learn about the reasons which others have found to support conclusions (for example, why the angle sum of any triangle is 180 degrees) and they also need to engage in their own reasoning both when working on what Polya calls 'problems to prove' and 'problems to find'. These two sides are connected. Learning about the reasoning of experts should assist in fostering your own reasoning abilities; it should establish a feeling that mathematics makes sense and is not just a set of arbitrary rules; and more generally, it should demonstrate the uniquely deductive character of mathematics.

I will report on two related studies that are relevant to the question of how students in Year 8 learn about reasoning. The starting point for both these studies is an international study, the TIMSS 1999 video study, which analysed a random sample of Year 8 Australian lessons and compared them with lessons from six other countries. The video study (http://www.acer.edu.au/research; http://www.lessonlab.com/timss1999) revealed many positive features of Australian classrooms. However, the Australian mathematics lessons displayed a cluster of features which I call the 'shallow teaching syndrome' (Stacey, 2003): a predominance of low complexity problems, which are undertaken with excessive repetition, and an absence of mathematical reasoning and connections in classroom discourse. To give just one example, only 2 per cent of the problem solutions presented by teachers or students in the Australian lessons demonstrated 'making connections', i.e. showed some linking between mathematical concepts, facts or procedures.

The first study (Stacey & Vincent, 2009) examined the way in which textbooks present explanations of mathematical results. It is often reported that secondary teaching is dominated by textbooks, and so it was of interest to us to see the nature of the reasoning that they display and promote. The study's focus was on explanations of *why* important mathematical results are true, not explanations of *what* or *how* (e.g. What does NNW mean?, How do you make a stem-and-leaf plot?). These *why* explanations involve mathematical reasoning at its best.

In the second study, also carried out with Dr Jill Vincent, we interviewed about 20 mathematics education leaders around Australia to explore their responses to the notion of the shallow teaching syndrome and the place of elements of working mathematics (including reasoning) in classroom teaching. They were education department officers, mathematics association leaders and textbook writers. Although the sample was too small to draw firm conclusions, there were few obvious differences in responses by employment type, although the education department officers were more aware of system level initiatives and the daunting scale of the task of reaching all schools with in-depth assistance.

For the textbook study, we selected nine popular textbooks from four Australian states, and within that chose seven topics where there was a result of mathematical importance that needed some justification or proof. Examples include the angle sum of triangles, multiplication of two negatives, the area of a circle and the rule for division of fractions. For each topic and each textbook, we examined all the explanations of the result presented explicitly in the explanatory text or the associated electronic material devoted to that topic. The explanatory text typically occupied half a page, but sometimes only one or two lines. We asked the 20 mathematics education leaders whether they thought the amount of classroom reasoning had changed since the 1999 study. The introduction of better electronic resources was the only reason given more than once for suggesting that there might have been positive change.

The first observation from the textbook study is that mathematical results are established using a variety of different modes of reasoning. Most of the textbooks made some attempt to explain every rule rather than simply presenting 'rules without reason'. Textbooks, and good lessons, build an understanding of mathematical results by offering a range of 'didactic explanations', including but not restricted to age-appropriate versions of 'proper' mathematical proofs. The phrase *didactic explanation* does not imply a verbal demonstration provided by the teacher or textbook in a colloquially 'didactic' manner, but is intended to recognise that there are many useful explanations for students in addition to formal proofs. A didactic explanation may be evident through guided discovery, use of a manipulative model, a data gathering activity, or a teacher presentation.

Many textbooks provide more than one explanation for a result. While multiple mathematical proofs of a result are in a sense redundant (one good proof suffices to prove), in teaching it is beneficial to offer multiple ways of establishing the same result. Seven different modes of explanations were identified. In a few cases, results are proved by deduction using a general case, in a way that closely approximates standard mathematical proofs, although at a low level of formality. Deductive reasoning is also evident in other ways. Since students at Year 8 do not speak algebra fluently, deduction is often not from a general case, but from a special case that is intended to be general. So, for example, students learned that multiplying two negatives results in a positive by cleverly extending the 5 times table to negative integers. Such expectation that students will see the general in the particular is very common in all mathematics teaching (e.g. demonstrating how to carry out an algorithm), but the textbooks did not draw any attention to the need to think of the specific case in a general way. This is one simple way in which students' appreciation of the unique features of mathematical reasoning could be improved, even before they have the formal mathematical language to deal with it well.

Didactic explanations using inductive reasoning that is more appropriate to science than mathematics, are common. Sometimes a rule is confirmed by showing that in specific instances the rule would give the same result as could be predicted from a model (for example, the result of sharing a quarter of a pizza between three people could be shown to be the same as the answer obtained by following the to-be-learned rule). At other times, students measure or count to empirically discover a rule from data, such as the angle sum of a triangle is 180 degrees. In a few instances, the textbooks made it clear that testing a few cases was not an adequate mathematical proof, but this could certainly be done more often to improve student awareness of reasoning. Many of the empirical activities seem to us to have substantial pedagogical value (as noted above, having multiple methods adds to learning), but textbooks could comment that their role is in mathematical discovery rather than in proof.

In some cases, the 'explanations' made no contribution to developing mathematical thinking at all. Sometimes, there was simply a statement or appeal to authority (e.g. *Euclid* or a computer), and others discussed loose qualitative analogies which may have had some mnemonic value but were not modelling the mathematical essence.

Looking over the results, it was clear that these textbooks generally paid reasonable attention to mathematical reasoning in explanations, and it is does not seem that prevalence of 'textbook' teaching is an adequate explanation for the lack of reasoning evident in Australian classrooms in the video study (although related factors such as a prevalence of low complexity problems in the textbooks certainly contribute). However, apart from offering examples of reasoning, there were few instances of instruction in mathematical reasoning. Amongst the 69 instances examined, one exception was that two textbooks explicitly rejected measuring for finding the angle sum of a triangle in favour of a deductive proof. In the other exception, a textbook mentioned that an explanation presented for a specific case could also be applied in all other cases, explicitly pointing to the generality that was required. Attention to instruction in reasoning, and to pointing out key elements of reasoning, would enrich the didactic explanations given.

We found that the nature of the reasoning depends on the result being explained. All textbooks had at least one deductive explanation of the formula for the area of a trapezium, but only half contained deductive explanations for the angle sum of a triangle. The nature of the reasoning also varies from textbook to textbook since different books are written with different student audiences in mind. In the interview study, one of the most common explanations for all features of the shallow teaching syndrome was the difficulty of providing suitable material of this nature to a mixed ability class. Overcoming this difficulty is not as simple as some people claim.

In the textbooks, explanations were generally very curtailed and usually omitted basic reasoning (for example, stating that a finding about a specific case also applies in general). Hence the explanations are unlikely to stand alone, and students must rely on teachers to elaborate. It is unlikely that all teachers can present these elaborations from the material provided, so this finding further highlights the often cited need for teachers to possess sufficiently strong mathematical knowledge and deep mathematical pedagogical content knowledge. This highlights another strong theme of the interview study, where many of the respondents expressed strong concern that teachers teaching out-of-field needed considerably more support to do a good job on the working mathematically themes.

For establishing a firmer place for mathematical reasoning in Australian classrooms than it has at present, I suggest the following.

1. Although all aspects of working mathematically are taught during engagement with the content of mathematics, this does not mean that they should not ever receive explicit attention. This applies at the level of classroom tasks, classroom discourse, unit planning and curriculum description. In classroom teaching, as in the textbooks, there are many opportunities where instruction in reasoning is simple to add.
2. A description is needed of a developmental path in mathematical reasoning across the grades, that would give teachers, textbook authors and curriculum writers a sense of what type of reasoning they can expect and encourage at each level and in what directions students' reasoning should be developed. This could not be as specific as in the content strands, but it could still be helpful in developing a shared vocabulary, clear goals and expectations.
3. Guidance for teachers be provided on the usefulness of didactic explanations, the distinction (in some cases) with age-appropriate proof, and ways of evaluating them.

4 The major purpose of explanations in the textbooks seemed to be to derive a rule in preparation for using it in the exercises, rather than to give explanations that might be used as a thinking tool in subsequent problems. Changing this practice could give reasoning more prominence.

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