

Australian Council for Educational Research

Teaching Mathematics?

Make it Count:

What research tells us about effective teaching and learning of mathematics

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Reconceptualising Early Mathematics Learning

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Teaching and Learning Early Mathematics in the 21st Century

- A more holistic view of the complexities of mathematics learning
- Numeracy priority at international level
- Early exposure to technologies influences informal notions of mathematics
- Research evidence: young children are capable of abstract mathematics
- Emphasis on development of deep and flexible mathematical thinking
- Focus on early data exploration, modeling and algebraic reasoning

Research Agenda

- Describe early bases of mathematical thinking: **Growth** of structural development for individuals
- Track 'dynamic' structural development for individuals within 'social/pedagogical' contexts
- Design learning opportunities to promote structural development (learning trajectories, pedagogical and assessment frameworks)
- An integrated perspective: **holistic view of common** ideas and processes

“The power of mathematics lies in relations and transformations which give rise to patterns and generalisations. Abstracting patterns is the basis of structural knowledge, the goal of mathematics learning”

(Warren, 2005)

Why do some children fail to develop mathematically?

Some children go through their entire schooling without learning any real mathematics because they do not abstract ideas in a way that promotes mathematical thinking ... pattern, structure and relationships – that's the essence of mathematics.

Awareness of Mathematical Pattern and Structure

“An Awareness of Mathematical Pattern and Structure (AMPS) generalises across early mathematical concepts, can be reliably measured, and is correlated with mathematical understanding”

(Mulligan & Mitchelmore, 2009)

Integrating Research Perspectives: Structural Development in Number, Space and Measurement, Data and Algebra

- Structural development of representations and problem-solving (Diezmann et al., 2002; Goldin, 1994; Mulligan & Watson, 1998; Thomas, Mulligan & Goldin, 2002)
- Combinatorial thinking/ analogical reasoning (English, 1999; 2004)
- Early patterning (Papic; 2007; Fox/Waters, 2007; Tzekaki, 2008; van Nes, 2009)
- Early algebraic thinking (Blanton & Kaput, 2004; Brizuela & Earnest, 2004; Carraher et al. 2007; Mason et al., 2009;; Schliemann et al., 2005; Warren, 2004; Warren & Cooper, 2009)
- Measurement e.g. 'Measure up' (Slovin & Dougherty, 2006), units of length/area (Outhred & Mitchelmore, 2000)

Integrating Research Perspectives: Early mathematics programs

- 'Building Blocks'
- Big Maths for Little Kids
- Curious Minds (The Netherlands)
- Luvverhulme Project
- Measure Up
- Math Recovery
- Count Me In Too
- First Steps Numeracy
- Project Good Start

Integrating Research Perspectives: Spatial Structuring

- Imagery and processing (Arcavi, 2005; Gray, Pitta & Tall, 1996; 2000)
- Spatial visualisation: pattern imagery (van Nes, 2009; Pirie & Kieren, 1994; Presmeg, 1998)
- Spatial structuring (Battista, 2003; Clements & Sarama, 2003)
- Structuring data: visualising data (Lehrer & Schauble, 2005)

Background Studies (Macquarie projects)

- Multiplicative reasoning and word problems
(Mulligan & Mitchelmore, 1997; Mulligan & Watson, 1998)
- Imagery and number concepts/ representations (Mulligan et al. 199,2000, 2002)
- Measurement units (Outhred & Mitchelmore, 2000; Outhred & Bragg, 2004; Curry et al., 2004)
- Structure of counting and number system (Thomas, Mulligan & Goldin, 1996; 2002)
- Count Me In Too/Counting On Numeracy framework(Mulligan & Wright, 2000; Thomas & Mulligan, 1999).

Pattern and Structure Projects 2002-2010

Development of PASA Instrument:
Assessment of multiple case studies:
baseline data: Grade 1 (n =103)
Interview-based assessment (Mulligan et al
2003;2005;2006;2009)

Single case studies (n =16) 'high' and 'low'
achievers
(Pre-posttest/posttest interviews)

Professional Development Project K-6
(1 year)
Pre-posttest design with intervention
(Mulligan 2006)

Intervention Study Kindergarten
10 case studies
Researchers as teachers

Evaluation study Kindergarten
(4 classes x 4 schools)
Experimental study/intervention

Studies on Patterning

Intervention Study Patterning
Preschoolers:
pre-posttest-delayed posttest with
intervention (Papic, 2005;2007; 2009)

Intervention study Patterning
Preschoolers: follow up studies +
PD model +framework (Papic, 2007-)

Design Study/Teaching
Experiment: PASMAT Grade 1
(over 2 years)
Teacher as
researcher(Crossing/Mulligan)

PD program: patterning-
Indigenous preschools
12 NSW communities 2007-
2010 (Papic)

Classroom action research 3D patterning
(Mc Knight, 2009) Grade 1

Broad Research Questions

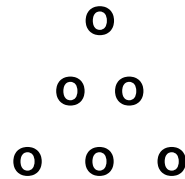
- Is 'pattern and structure' a general underlying construct that is generic to all mathematics learning?
- Do young children continue to develop and use structure consistently across mathematical content domains and contexts over time?
- Do all young children progress through identified stages/levels similarly?
- Why do low achievers fail to recognise pattern and structure in simple mathematical situations?
- How can a teaching and learning program be developed to promote pattern and structure?

Research Findings

- Young children's use of mathematical pattern and structure generalises across mathematics content areas (Mulligan et al., 2004; 2005; 2006)
- Early school mathematics achievement linked with development and perception of pattern and structure
- Awareness of pattern and structure can facilitate learning of, and connections between, key mathematical processes (Mulligan et al., 2006)
- Students experiencing difficulty in learning mathematics do not recognise pattern and structure; increasingly confused and 'crowded' mathematical ideas and representations (Mulligan et al., 2005)
- Awareness of Mathematical Pattern and Structure (AMPS) can be described as a general underlying construct (Mulligan & Mitchelmore, 2009)

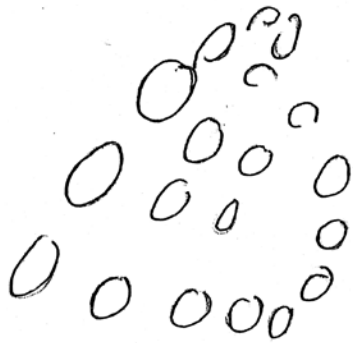
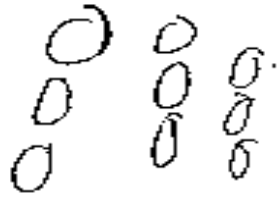
Triangular Pattern Task

(Flash card with pattern) Look carefully. Cover. Draw exactly what you saw. Describe it.



Triangular Pattern: Responses Over Time

School LA Student Interview 1
Lorian B



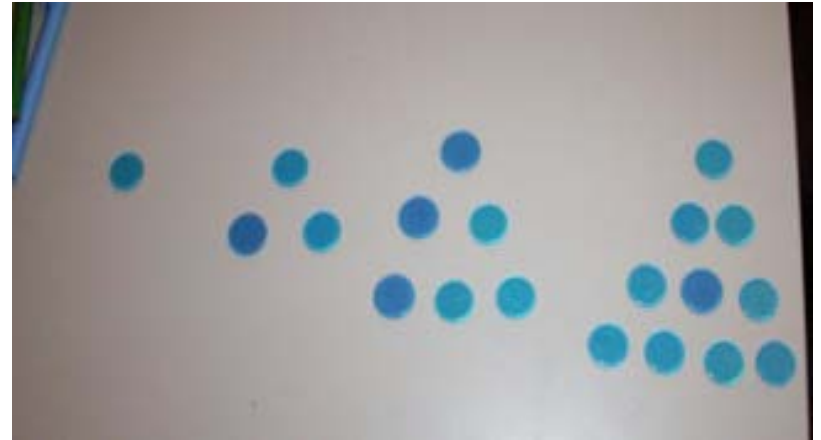
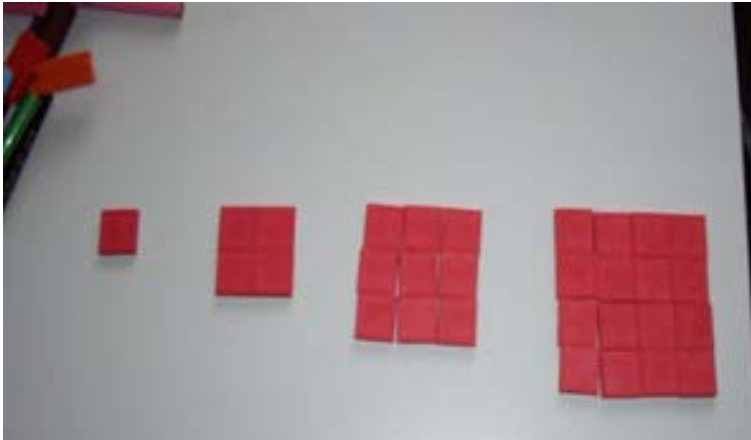
Research Findings: Patterning Studies (Papic)

- Preschoolers capable of simple and complex repetition, border and spatial patterns
- Growing patterns emerged with children who were competent at simple repetition
- Unit of repeat critical to development: link to multiplicative thinking early
- Framework explicitly supported professionals
- Developmental sequence identified: copy, create, extend; identify unit of repeat: Represent from memory, transfer pattern to other situation, emergent generalisation
- Evidence supported by K intervention study (Mulligan et al 2008)

Growing Patterns

"You just keep adding a row every time to what you had before ... that's one bigger than before"

(4 year old)



Aims of the Kindergarten Evaluation Study 2008-2010

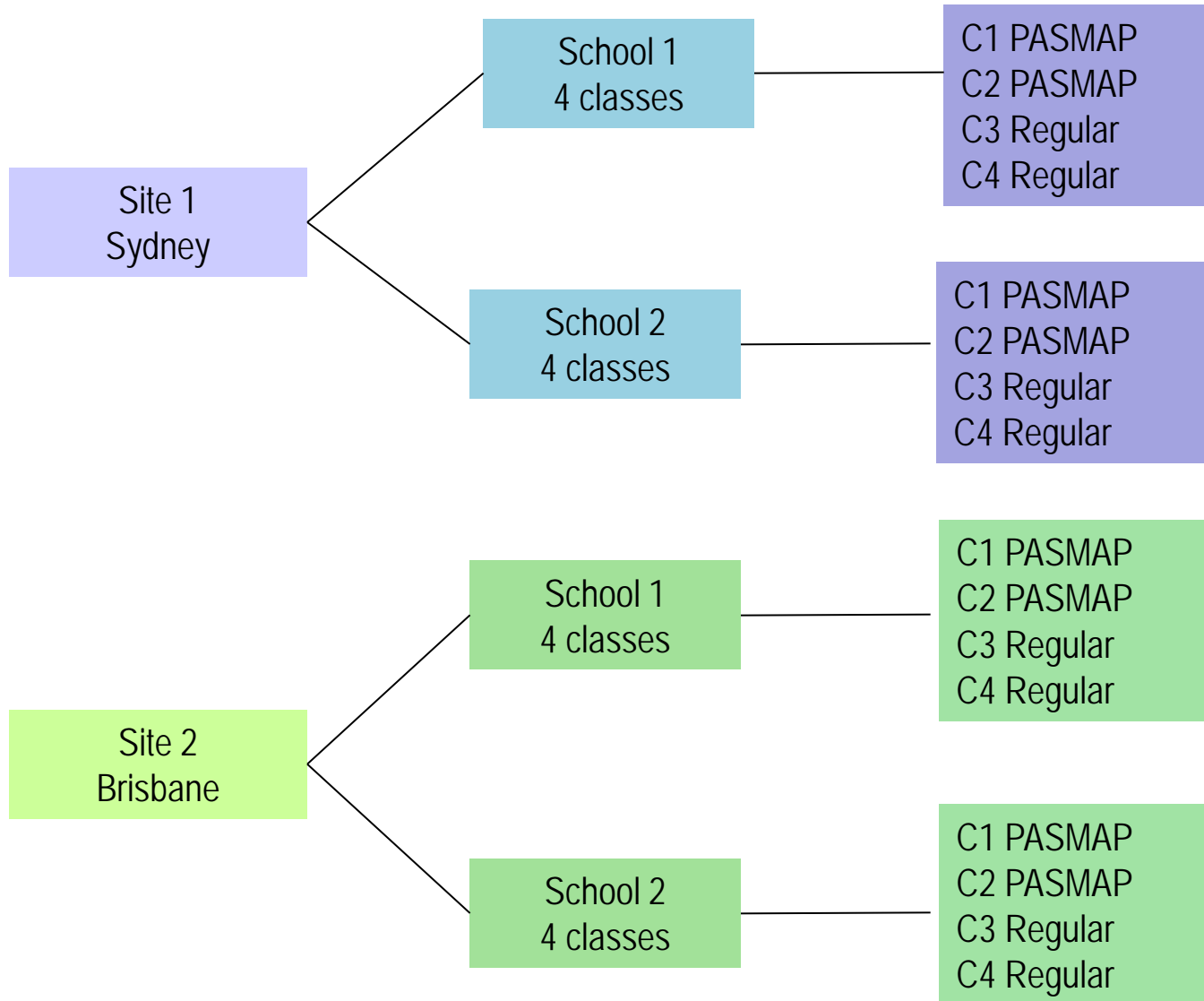
- To validate a new conceptual framework for mathematics learning based on the development of pattern and structure
- Evaluate the effectiveness of a school-entry mathematics program on learning
- Promote opportunities for deep mathematics learning
- Provide basis for describing early algebraic reasoning



Method

- Experimental evaluation study: Kindergarten classes from four large primary schools in Brisbane and Sydney (n=316)
- In each school:
 - two of four kindergarten teachers trial integrated PASMMap program
 - two continue with school's regular program
- Intervention implemented by teachers over three terms of Kindergarten
- Pedagogical framework developed and refined (learning trajectories)
- Narrative profiles of teachers and two target groups of five children within each of the four classes in each school compiled as case studies

Design



Stages of Structural Development

- **Pre-structural:** representations lack evidence of numerical or spatial structure
- **Emergent** (inventive-semiotic): representations show some relevant elements but numerical or spatial structure is not represented
- **Partial structural:** representations show most relevant aspects but representation is incomplete
- **Stage of structural development:** representations correctly integrate numerical and spatial structural features

(Adapted from Thomas, Mulligan & Goldin, 2002)

Professional Development



- Professional development was provided at initial stage
- Framework of project and learning sequences outlined with learning experiences, outcomes and core components
- Differentiated for individual teachers

Professional Development

- All teachers receive same amount of professional development support
- Experimental teachers focus on Pattern and Structure program
- Regular teachers focus on state syllabus
- Regular support, observation and de-briefing (once weekly)
- Impact on whole school development and change

Research Process 2009

- Feb: 'I Can Do Maths' assessment all K children. Ten case studies per class (5 high ability, 5 low ability) identified.
- March: Professional Development Program: K Teacher release one training day (funded), and other staff briefed.
- April: Pattern and Structure assessment (PASA) interview for all case study children.
- May- November: Maths lesson, outlined in PASMAMP framework, implemented daily* by teachers with weekly classroom-based assistance by researchers
- September 2010 ICDM, PASA and other assessment re-administered to case study children

* Wide variations between schools and classes in number of lessons and number of hours per week devoted to mathematics.

I Can Do Maths Assessment - Level A

- 30 Questions - written open-ended and multiple choice
- Administered orally to groups of up to 10 students
- 12 Number, 10 Measurement and 8 Space questions
- Year 1-level norms (ACER 2006) and analysis of strands

PASA Interview Assessment Items

17 items (6 verbal, 6 modeled and 5 drawn responses):

1, 2. Patterning: simple & complex repetitions

3. Subitising

4. Counting by twos and threes

5. Recognising ten as a unit

6. Fractions: halves and thirds

7. Visual memory: 3x4 grid

8. Volume: units / spatial

9. Combinatorial: multiplication

10. Quotition / division

11. Analogical reasoning

12. Transformation

13. Visual memory: dot pattern

14. Functional thinking

15. Time: clockface

16. Area: unitising

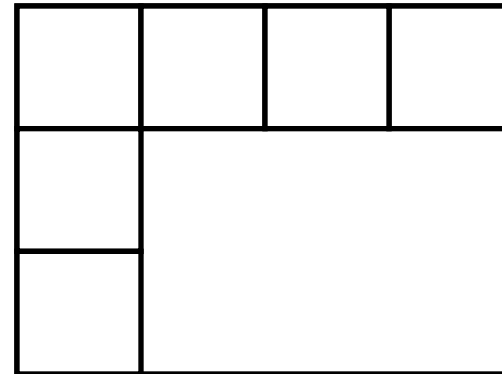
17. Volume: unit comparison

Unitising Area

Provide diagram on sheet, pencil, eraser. Record student code and date.

Someone has started to draw in some squares to cover this shape. Finish drawing the squares here.

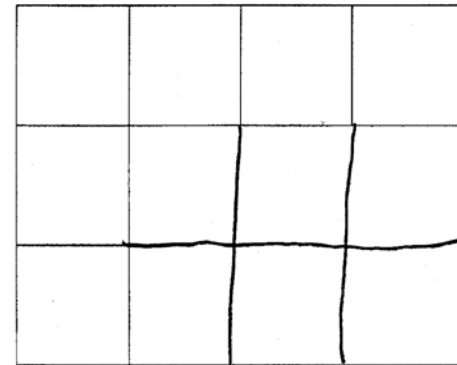
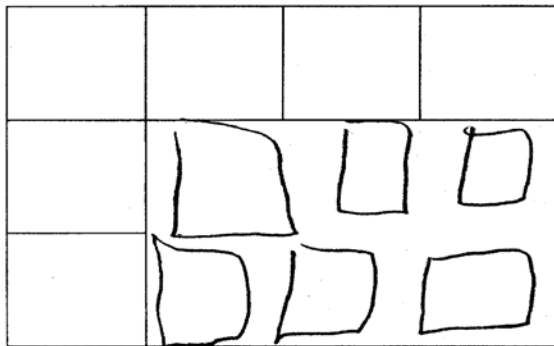
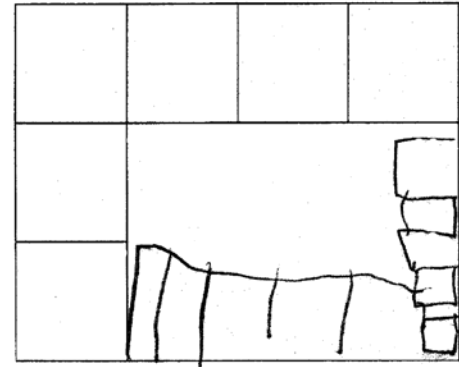
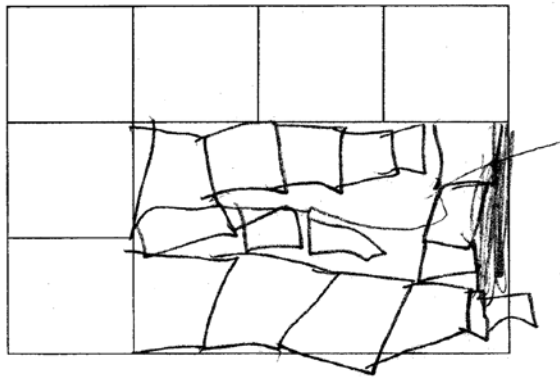
Point to the space.



Development of Units of Area

("Finish drawing these squares to fit into the rectangle")

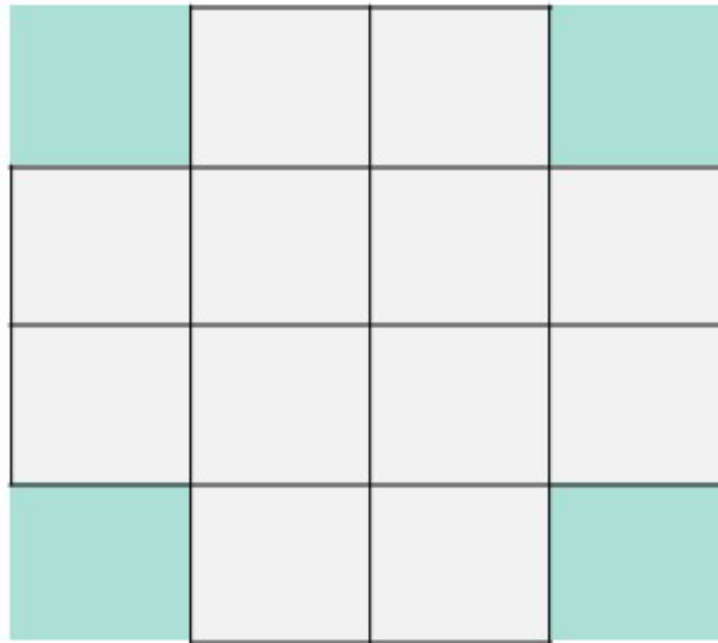
(Outhred & Mitchelmore, 2000)



Space and Volume

Imagine this shape folded up to make a box.

How many cubes would fill this box (without any spaces left)?



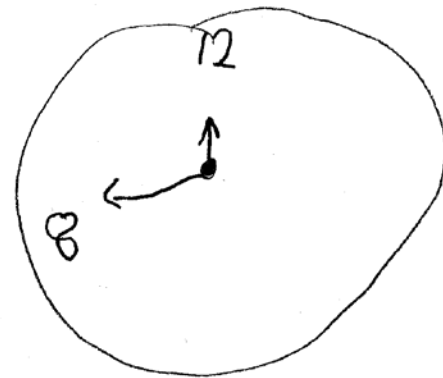
Analogical Reasoning

Your hand goes with your arm in the same way as your foot goes with your ...

Interviewer points to own hand, then arm then foot as he/she says the word. Do not point to leg.

Structuring Time: Clockface

("Draw the clockface so I know it is 8 o'clock")




Regular Program Characteristics

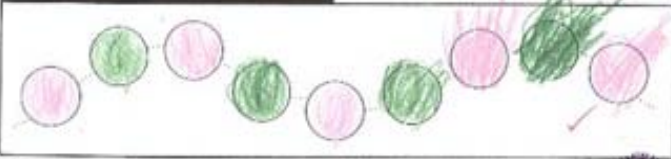
- Textbook or worksheets followed sequentially to meet pre-planned yearly schedule
- State syllabus matched to school program and or textbook or worksheets
- Lessons usually followed teacher directed initial sequence followed by activity and recording
- Limited differentiated planning or activities
- Teachers planned and evaluated independently of PASMAT teachers

Examples of Regular Program

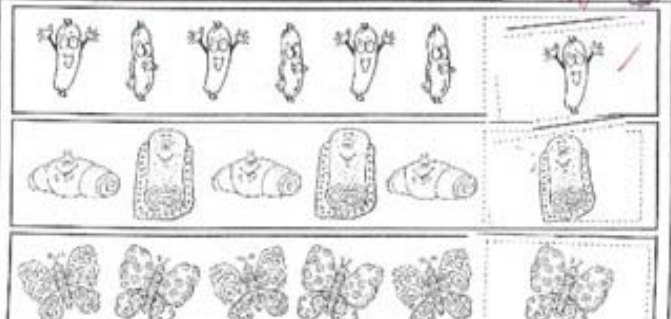
Copy the pattern.



Use two colours to make a pattern.



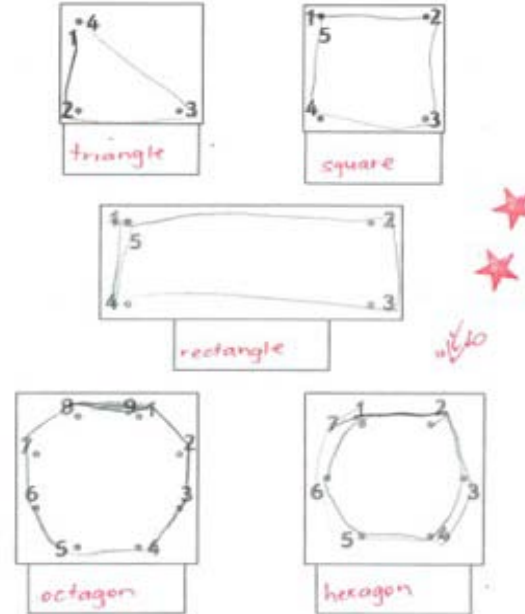
Cut out the shapes and paste them to make a pattern.



Linear patterning

Connect the dots to draw the shapes, and then label each shape:

hexagon square rectangle
octagon triangle





triangle square
rectangle octagon hexagon


Drawing shapes

Examples of Regular Program



Use the pictures to finish the number sentences:

1  **2** 

2 rows of 3 = 6 3 rows of 4 = 12

3 

2 rows of 10 = 20

4  **5** 

12 rows of 1 = 12 3 rows of 2 = 6

or 6 groups of 2 = 12 or 2 groups of 6 = 12

Using arrays

Off to the City

Draw something that is:

 taller ✓	 wider ✓
 lower ✓	 shorter
 narrower ✓	 longer ✓

Measurement

PASMAP Approach

- Highlight pattern and structure in scaffolded learning sequences
- Draw attention to mathematical features – “sameness” and “difference”
- Explicit focus on one aspect of structure at a time
- Encourage connectedness between structural features
- Visual memory activities: recording from memory
- Representations constructed by student
- Explain and justify: encourage simple symbolisation and emergent generalisation
- Measurement and spatial structuring as a basis for number concepts: unitising as a common process

Key Components of the PASMAM Intervention

- **Counting:** Counting with shapes, staircase patterns, empty number line, numeral tracks
- Subitising followed by partitioning and extending dot patterns: grouping/estimating
- Rhythmic and perceptual counting
- Counting as linear measure
- Structure of base ten: counting patterns/ten frames/100 charts
- **Patterning:** Simple (AB) and complex (AAB) patterns - with and without models
- Growing patterns: focus on development of pattern of squares
- Border and non linear patterns
- **Spatial Properties:** Similarity and congruence (2D shapes)
- Symmetry and transformations

Key Components of the PASMMap Intervention

- **Partitioning**/ part/whole relations/ fractions
- Spatial structuring area, co-linearity: Grids
- Equivalence and commutativity: links to multiplicative

- **Unit of repeat** linked to composite units
- Equal units; co-linearity, commutativity: Arrays

- Measurement concepts : **structure of the unit** (length, area, volume, mass)

Examples of PASMMap Program:

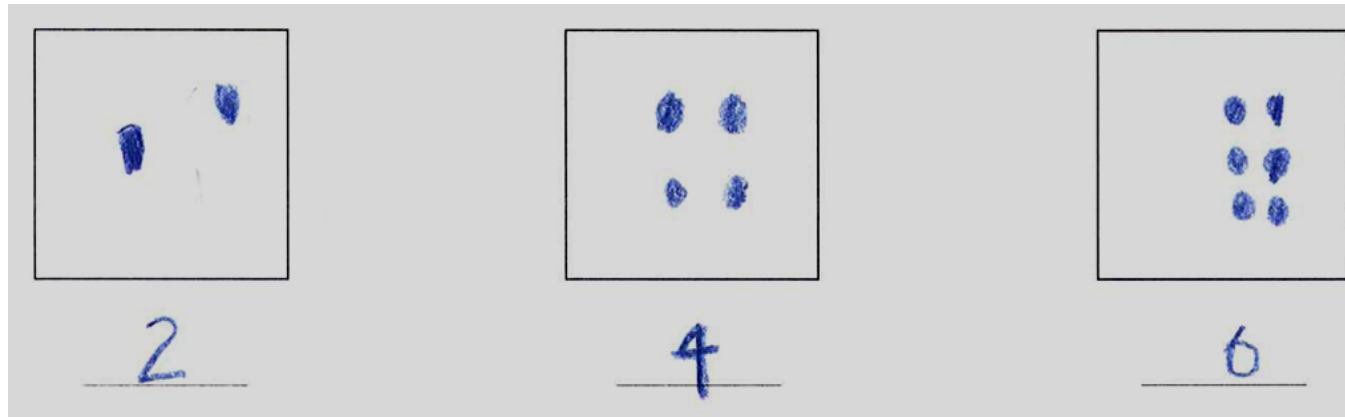
Structuring Counting and Grouping



- Counting as a unit of repeat
- Using the unit as a measure
- Focus on counting in patterns and multiples rather than unitary counting

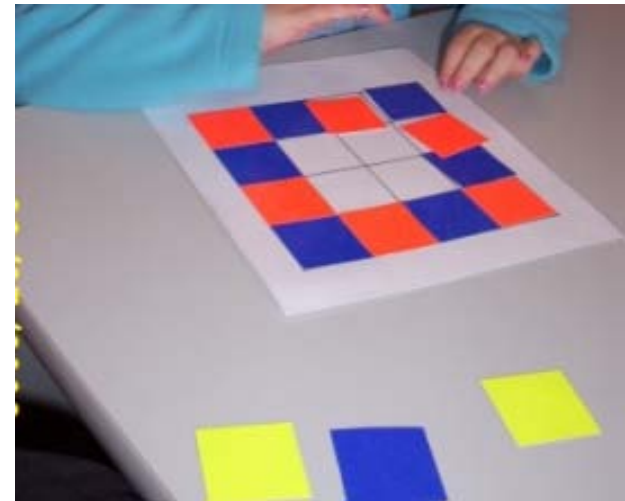
Subitising

Recording subitising dot patterns

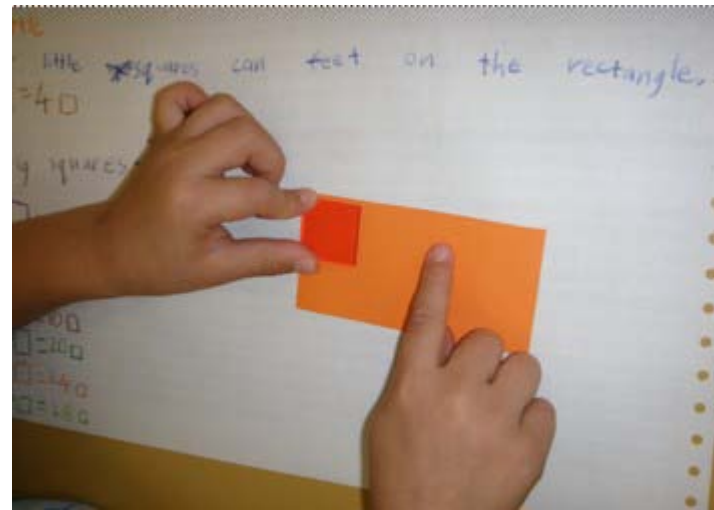
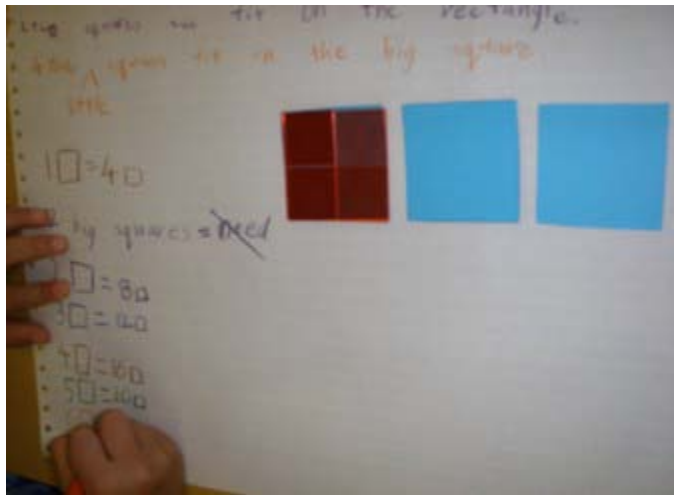


Patterns using Borders

- Complete a 4 x 4 AB border pattern using cut-out tiles and grid template. Extend to 5 x 5 grid and 4 x 6
- Identify unit of repeat
- Focus on multiplicative idea "my pattern is repeated 6 times"



Early Multiplication: Using four squares (grid structure) as a composite unit



"I made a pattern so 1 big square is 4 little squares so its 4 for each square and 6 or 8 for rectangles. Every time you use the square it's a four"

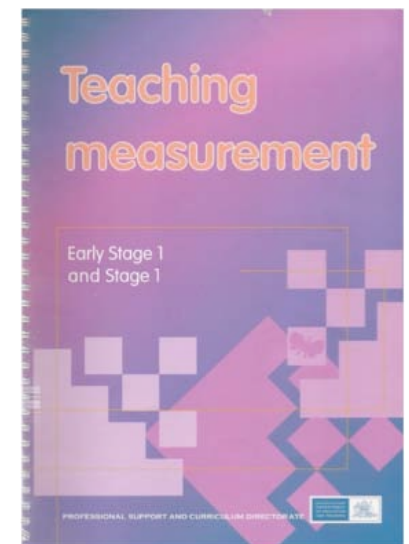
Measurement: 'Structure' of Concepts

- "The smaller the unit, the more you will need to measure"

Jane's scoop is twice as big as Peter's scoop to measure 1kg of rice

- The more (equal) parts, the smaller the fraction (and the larger denominator)

Which is smaller $1/5$ or $1/6$?



Structuring Relationships: Equivalence

$$1 + 2 = 2 + 1$$

$$1 + 3 = 3 + 1$$

$$1 + 4 = 4 + 1$$

$$1 + 5 = 5 + 1$$

$$1 + 6 = _ + 1$$

What is the same/different? How many different patterns can you find?

Make another pattern with the same structure.

Forms of Analysis

- Program descriptors and framework

Each PASMMap focus student tracked:

- Developmental pathway of mathematical concepts and processes
- Quality and growth of the underlying structural characteristics
- Evidence of relationships built between concepts and processes
- Evidence of emergent generalisations

Teachers' views of the impact of the program on student learning: video data, journal record, interviews (individual and group)

Quantitative analysis:

- Comparison of ICDM and PASA scores for PASMMap and non-PASMMap students: state, group, grade, ability level, gender differences
- PASMMap responses coded for level of structural development
- Other measures at delayed posttest

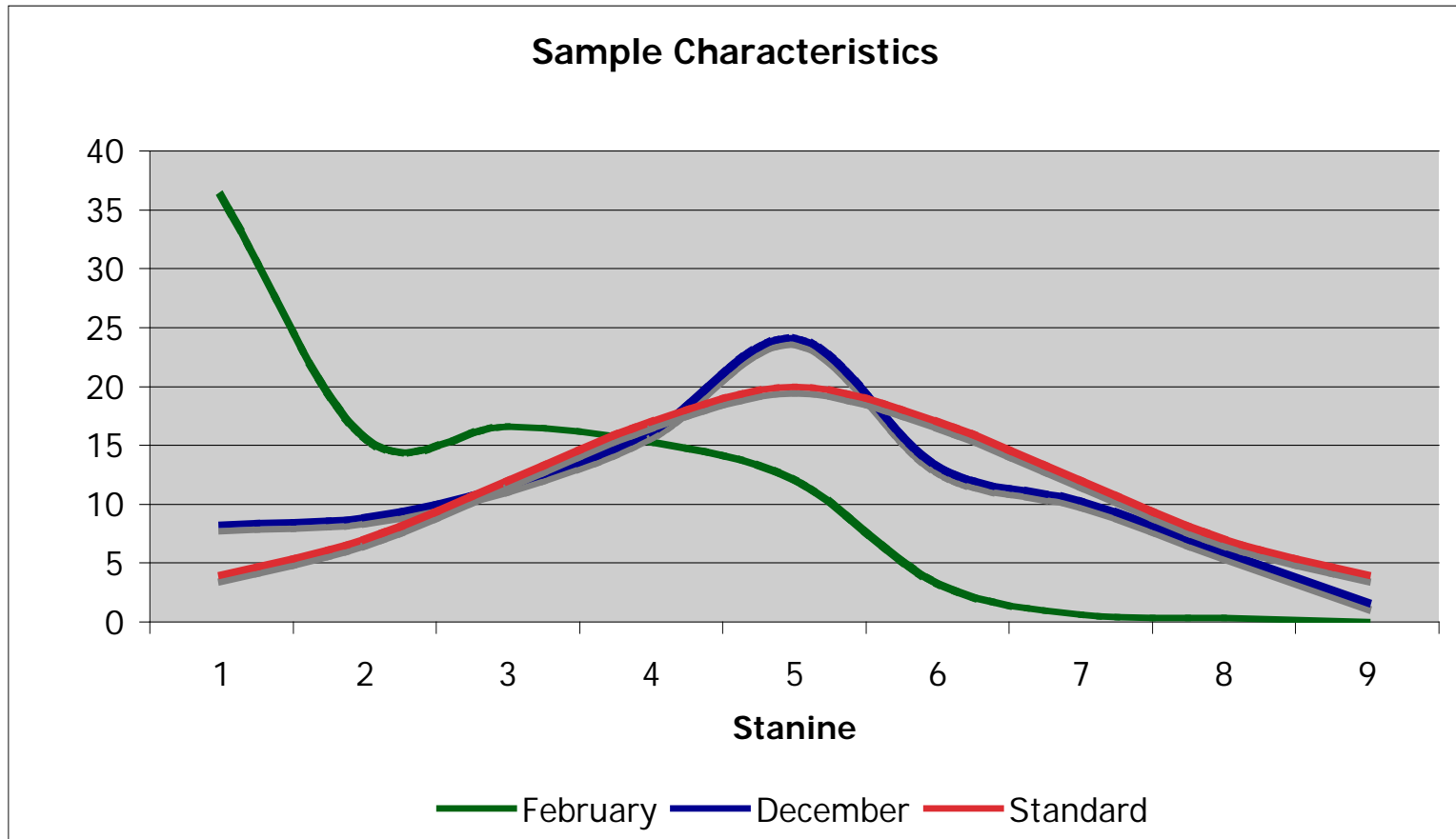
Preliminary Findings

Quantitative analysis ICDM

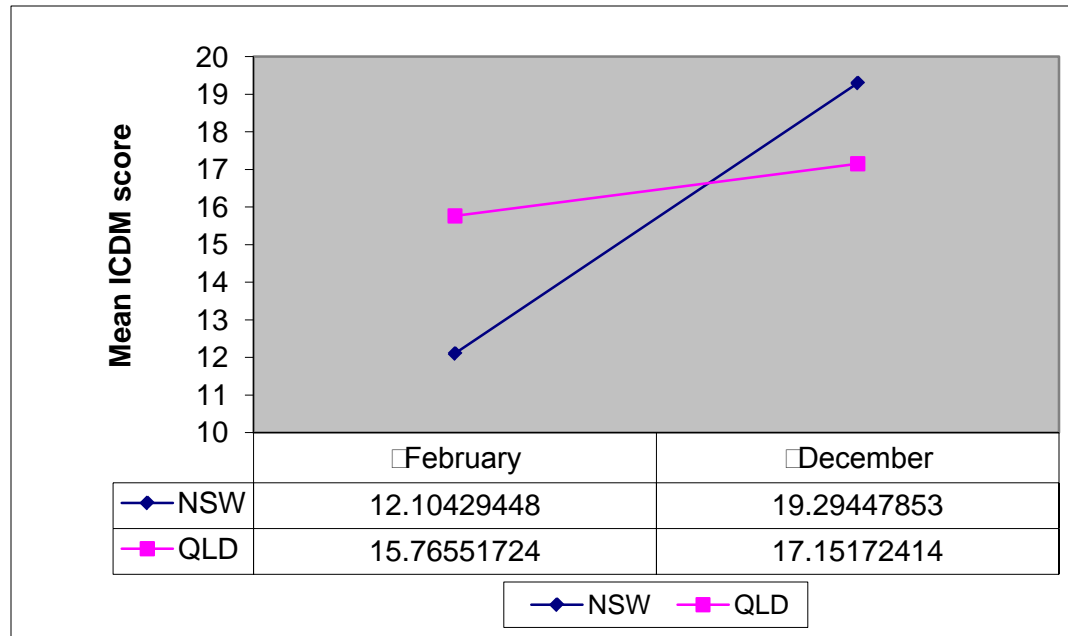
- Significant differences between states (NSW and Qld)
- Significant differences between classes (teachers)
- Both groups showed impressive growth consistent with Year 1 level norms (ACER, 2006)
- PASMMap students showed higher mean scores but no significant differences found between PASMMap and regular students on ICDM at post-test.

ICDM Assessment (Sample)

Comparison of whole sample (n=316) with Year 1 norms (2006)



ICDM Level A Pre- and Post-test by State



Preliminary Findings

Teacher/Classroom data

- Stark differences between PASMMap and Regular teachers' implemented program scope and depth and pedagogical approaches
- All PASMMap teachers adhered to the planned PASMMap program sequence but wide variations in time allocated
- Initial concerns that the state and school "requirements" would not be met were gradually alleviated
- Classroom data (video of teaching episodes) shows increased confidence, PCK and independent thinking by PASMMap teachers
- Hightened awareness of all teachers to focus on implementing a consistent maths program

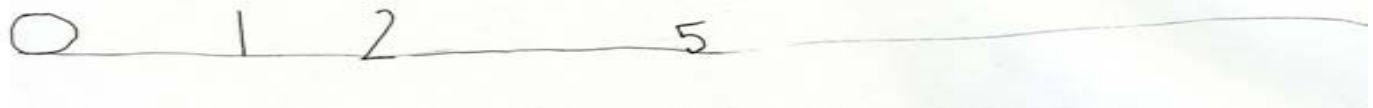
Teachers' Views

Preliminary Findings

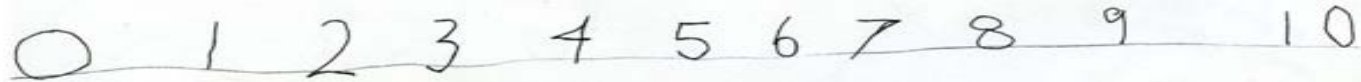
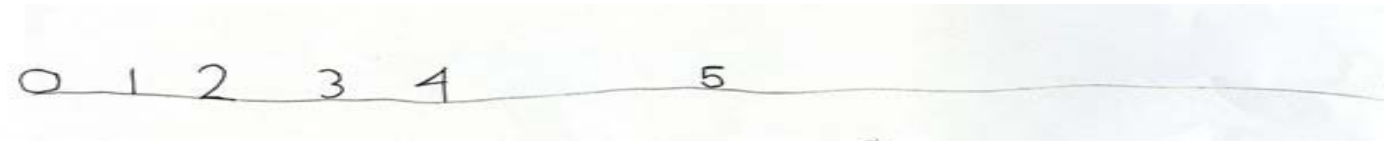
Individual 'focus-student' data (classroom based)

- PASMMap students demonstrated abstract and creative mathematical ideas and formed emergent generalisations
- Engaged and for longer periods of time
- Stark differences between PASMMap and Regular students' process of learning: PASMMap approach focused on similarity and difference, connecting each aspect of the learning sequence
- Highly able PASMMap students developed systems that had common structural features including visual memory tasks for each lesson
- PASMMap High ability group were most advantaged by the opportunities
- Regular students demonstrated achievement of the expected outcomes in the same ways: no differentiation
- Most PASMMap students demonstrated learning outcomes well beyond Kindergarten curriculum expectations

Structuring Counting as a Unit of Measure: "Show 0-10 on the line as you count"



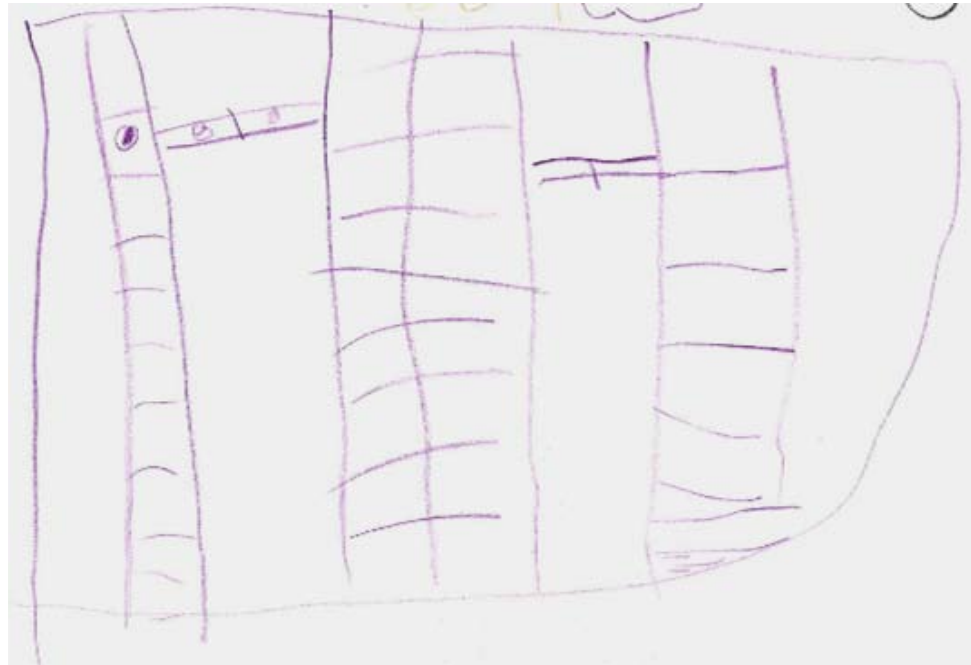
Shows half way as 5



"Oh, I didn't know that the numbers are the same space between even when you get a fat number, so 2 is 1 space bigger than 1, and 3 is one bigger along than 2. So I have to make the spaces the same size going along the line even if my numbers are getting bigger and you get numbers together like 99... so it doesn't matter if I count a long way, past 100, the space is the same long cause its one more every time.. Oh I get it now"

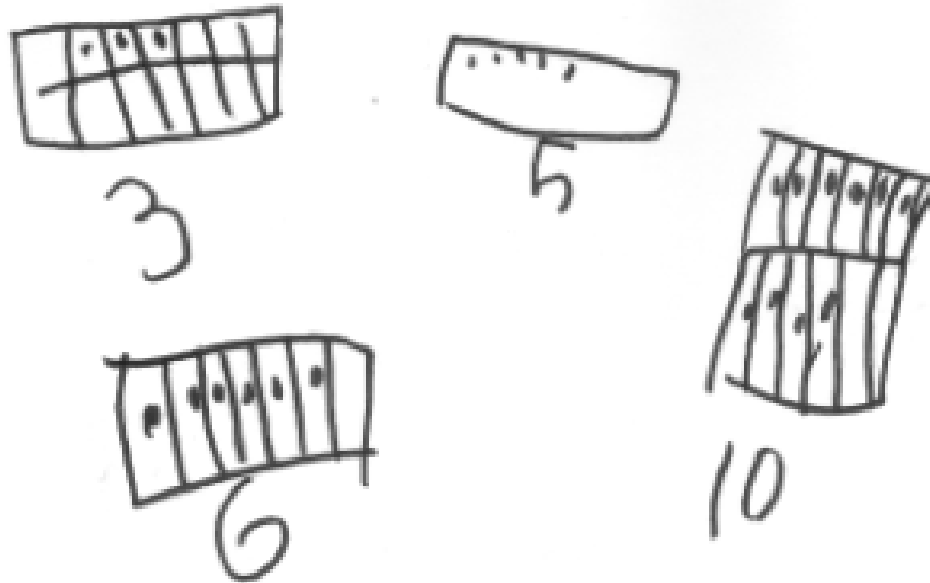
(Sasha, 6 years 4 months)

Constructing, Representing and Using Ten Frames from Memory



Pre-structural idiosyncratic image of “tall buildings with bridges”

Constructing, Representing and Using Ten Frames from Memory



Emergent structure: "single and double" frames

Constructing, Representing and Using Ten Frames from Memory



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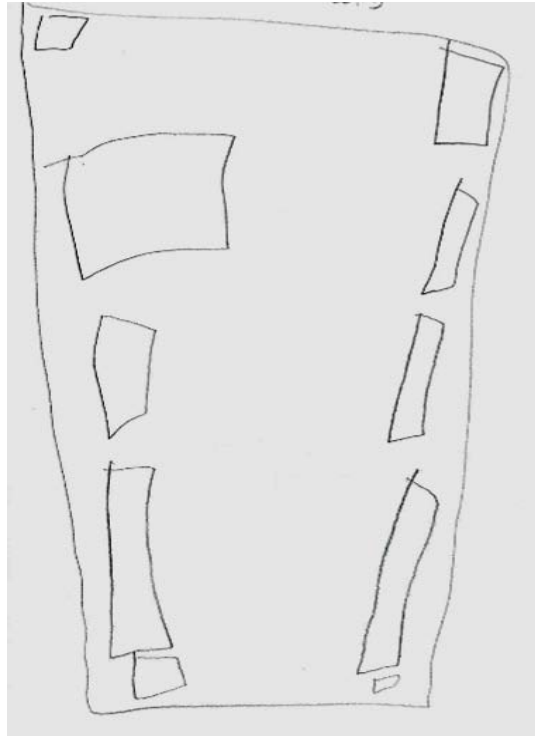
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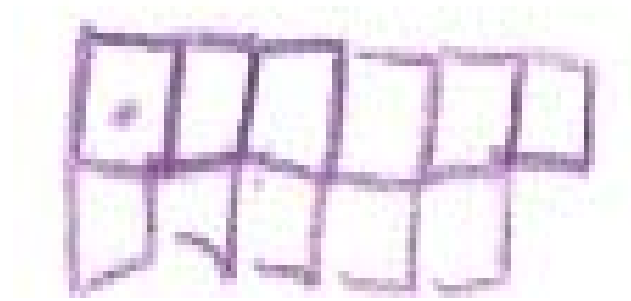
Emergent structure: single units

Constructing, Representing and Using Ten Frames from Memory



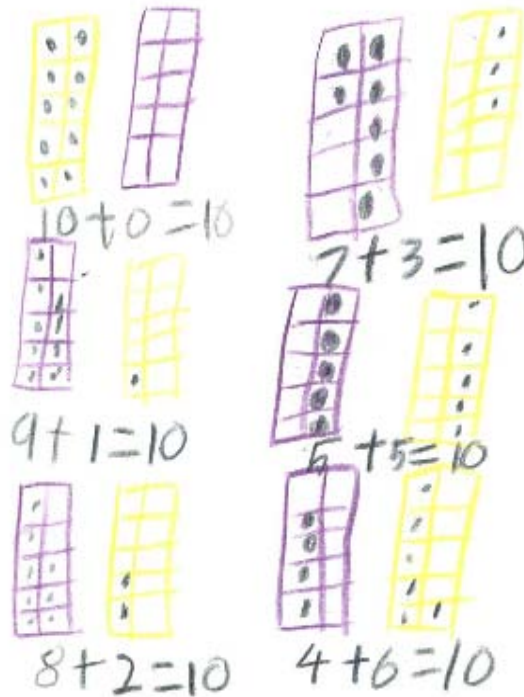
Partial structure: shown by 2x5 unequal units

Constructing, Representing and Using Ten Frames from Memory



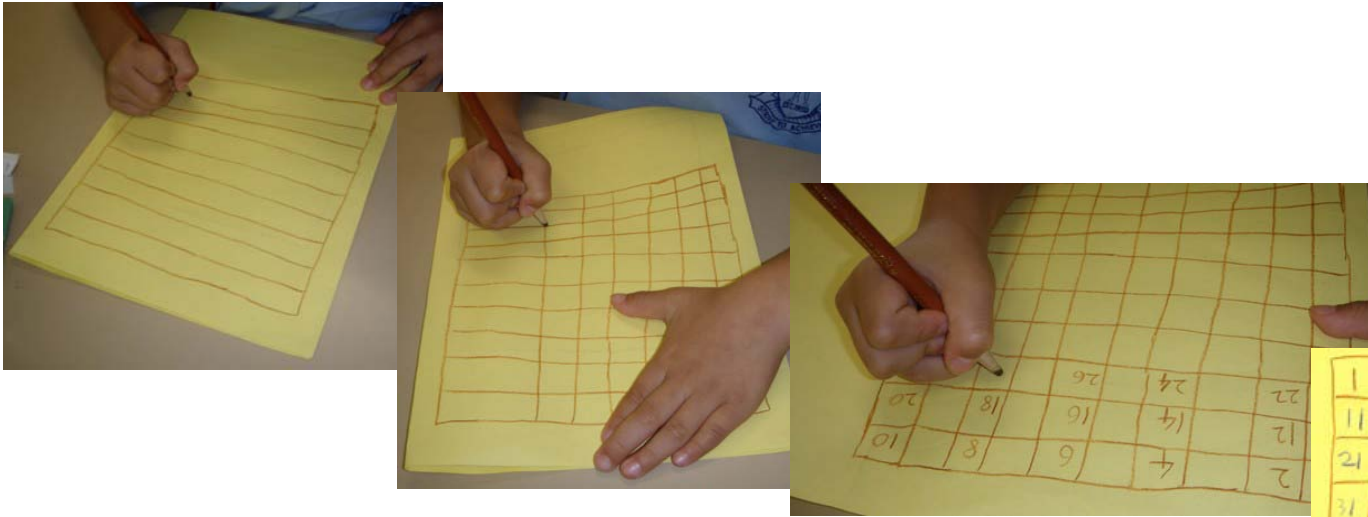
Partial structure: aligned single units in ten frame structure

Constructing, Representing and Using Ten Frames from Memory



Structural features using 2 ten frames to create number facts as a pattern, showing 5-wise pattern.

Constructing Hundreds Chart from Memory (structural response-emergent generalisation)



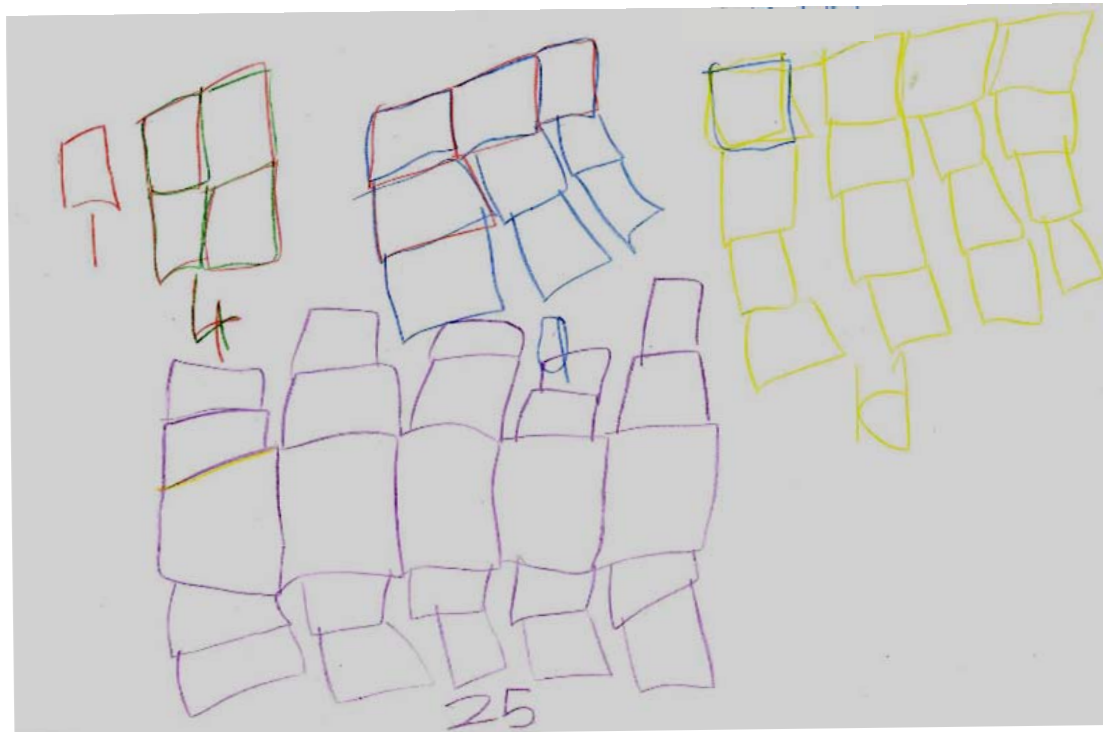
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

“We need 100 numbers, so 10 rows and 10 in each row. All numbers on the end in the column has to end with the same number....it doesn't matter how far I go with the numbers... cause it's the tens column we are adding...one more ten each time” (Jon, 5 years 8 months)

Constructing Hundreds Chart from Memory

Pre-structural to structural responses

Drawing and Explaining Pattern of Squares from Memory



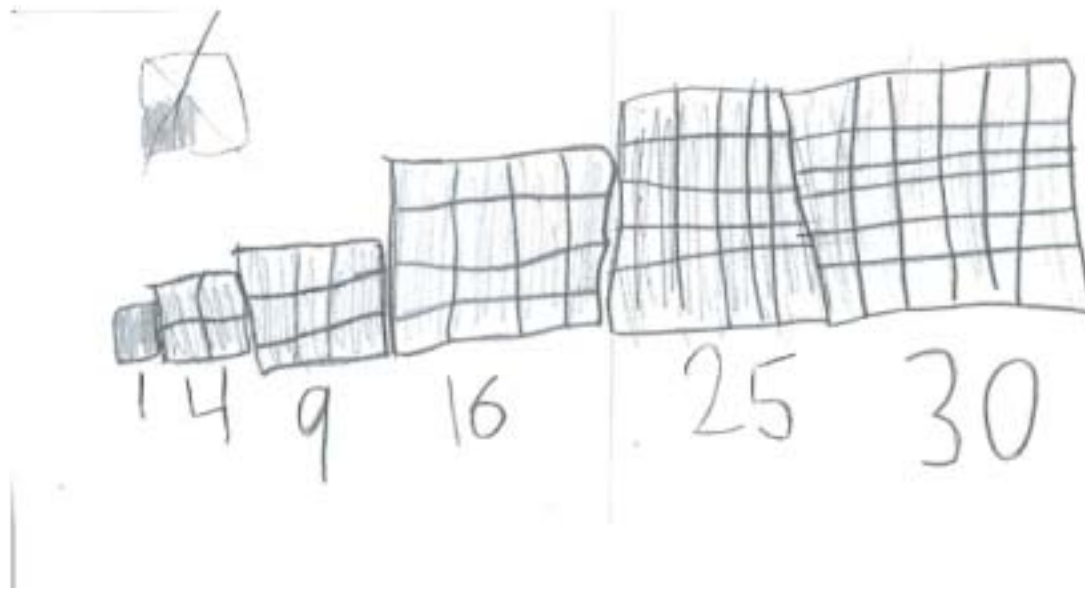
Emergent structure: pattern of squares using single units.

Drawing and Explaining Pattern of Squares from Memory



Partial structure: equal sized single units but lacks co-linearity and "growing square" properties.

Drawing and Explaining Pattern of Squares from Memory



Partial structure: pattern of squares limited to 5x5

Drawing and Explaining Pattern of Squares from Memory



Structural response showing pattern, square grid structure, pattern sequence

Limitations

- Wide differences in the time allocated to implementing the PASMMap program and the teaching of mathematics overall
- Half the PASMMap program components successfully implemented but program scope and effectiveness needs to be evaluated over longer period of time
- Teacher knowledge limited and much more professional learning support needed including initial training
- Use of standardised instrument to measure mathematics achievement limited
- New measures of pattern and structure need to be developed
- Professional support materials need to be comprehensive
- Program evaluation extended to more diverse samples of students and at higher grade levels

How can we promote pattern and structure?

- Provide experiences that challenge children's mathematical thinking and problem solving
- Assist children in seeing the underlying structure of different mathematical situations
- Provide a range of experiences in which children explore various patterns in numerical and spatial situations
- Provide activities that require children to interpret, describe, explain, question, analyse, critique, debate, justify, and predict.

Implications for Curriculum



“An algebraic perspective can enrich the teaching of number...and the integration of number and algebra, especially representations of relationships can give more meaning to the study of algebra in the secondary years. This combination incorporates pattern and/or structure and includes functions, sets and logic.”
Australian National Curriculum (ACARA, 2010)

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