



# **Mathematics teaching and learning to reach beyond the basics**

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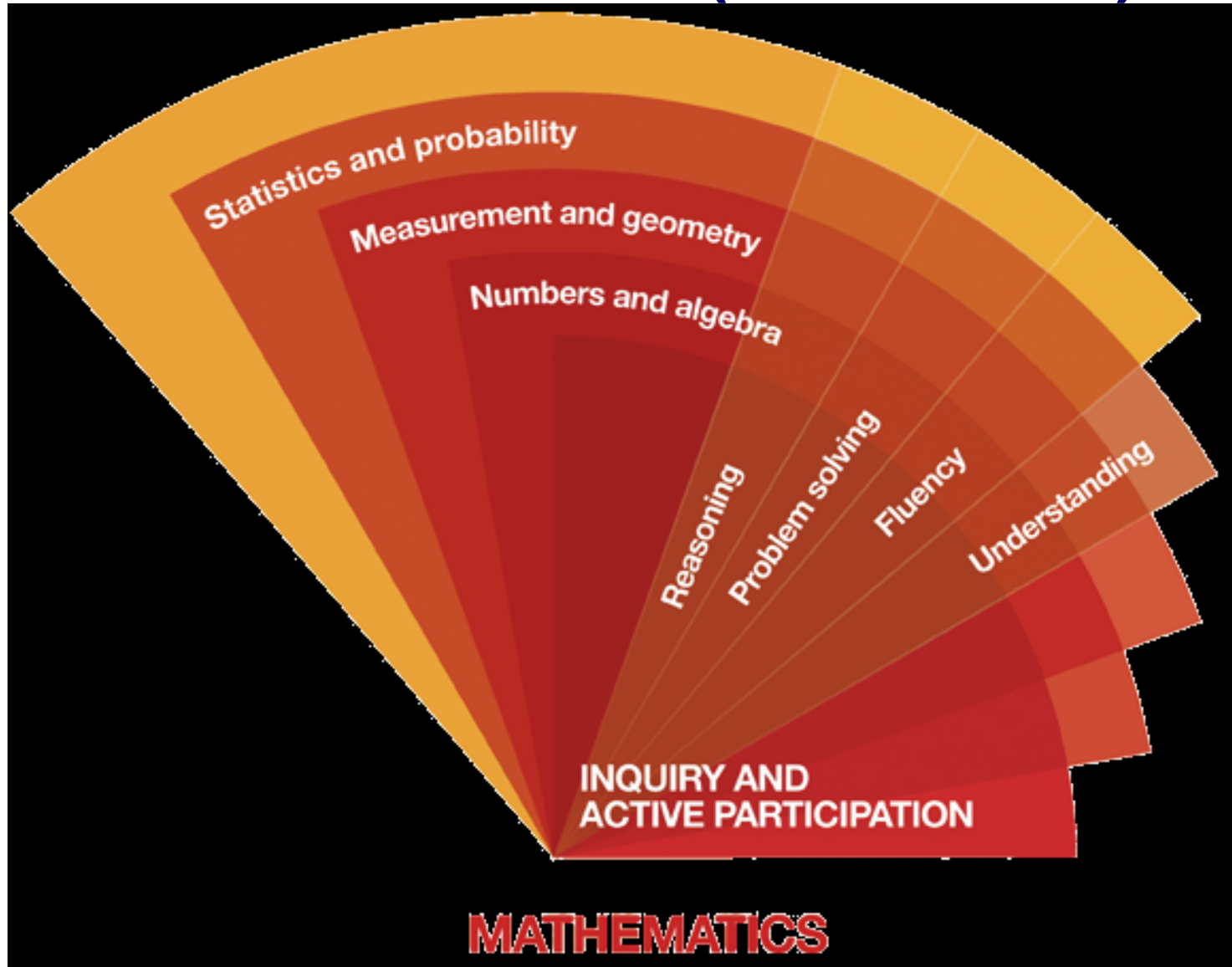
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# Challenge



- Encouraging mathematics learning that goes beyond ‘the basics’
  - Learning and teaching ‘the basics’ in mathematics is hard enough
  - Also need to aim for
    - Deep and robust understanding – not rules without reasons
    - Flexible problem solving
    - Ability to use mathematics
    - Knowledge of mathematics as a way of thinking
- Focus on mathematical reasoning
  - in classrooms
  - in the Australian Curriculum

# Australian Curriculum (2010 March)



# Australian Curriculum (2010 March)



- Proficiency Strands
  - Understanding
  - Fluency
  - Problem Solving
  - Reasoning

- Content Strands
  - Statistics and Probability
  - Measurement and geometry
  - Number and Algebra

*Australian Mathematics curriculum (March 2010)*

*Proficiency Strands*

## **Understanding**

Students build robust knowledge of adaptable and transferable mathematical concepts, make connections between related concepts and develop the confidence to use the familiar to develop new ideas, and the 'why' as well as the 'how' of mathematics.

## **Fluency**

Students develop skills in choosing appropriate procedures, carrying out procedures flexibly, accurately, efficiently and appropriately, and recalling factual knowledge and concepts readily.

## **Problem solving**

Students develop the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively.

## **Reasoning**

Students develop increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying, and generalising.



# Mathematics content and process

- 1970s: first explorations of mathematical modelling
- 1980s: NCTM agenda for action
  - promoting problem solving
  - by 1988, prominent in every state document in Australia
- 1990s: Working Mathematically strand (many variations)
  - strand includes problem solving, reasoning, communication, technology.....
- 2010: Proficiency strands
  - reasoning and problem solving
- ALWAYS: Want WM a reality in most classrooms – an elusive goal
  - TIMSS video study 1999 a reality check – “shallow teaching syndrome”

# Plan



- A little background on the “shallow teaching syndrome”
- Several studies looking at this
  - Survey of mathematics education leaders
  - Review of mathematics textbook problems
  - Review of the nature of reasoning and explanation in textbooks
- Video of classroom highlighting some aspects of reasoning
- Discussion of reasoning strand in Australian Curriculum



# SHALLOW TEACHING SYNDROME: Textbook Survey

# TIMSS Video Study 1999: Methodology



- Video-taped 638 Year 8 lessons from the seven countries
  - Video-taping spread across school year
  - Lesson selected at random; little warning given
  - Random sample of schools and (volunteer) teachers
- Australia: 87 schools, 1950 pupils
- Comparative analysis of many characteristics

Hiebert, J., Gallimore, R., Garnier, H., Givvin, K.B., Hollingsworth, H., Jacobs, J., Chui, A.M.-Y., Wearne, D., et al (2003). *Teaching Mathematics in Seven Countries: Results from the TIMSS 1999 Video Study* (NCES 2003-013). Washington DC: National Center for Education Statistics.

Hollingsworth, H., Lokan, J. & B. McCrae, B. (2003) *Teaching Mathematics in Australia: Results from the TIMSS 1999 Video Study*. Melbourne: ACER.

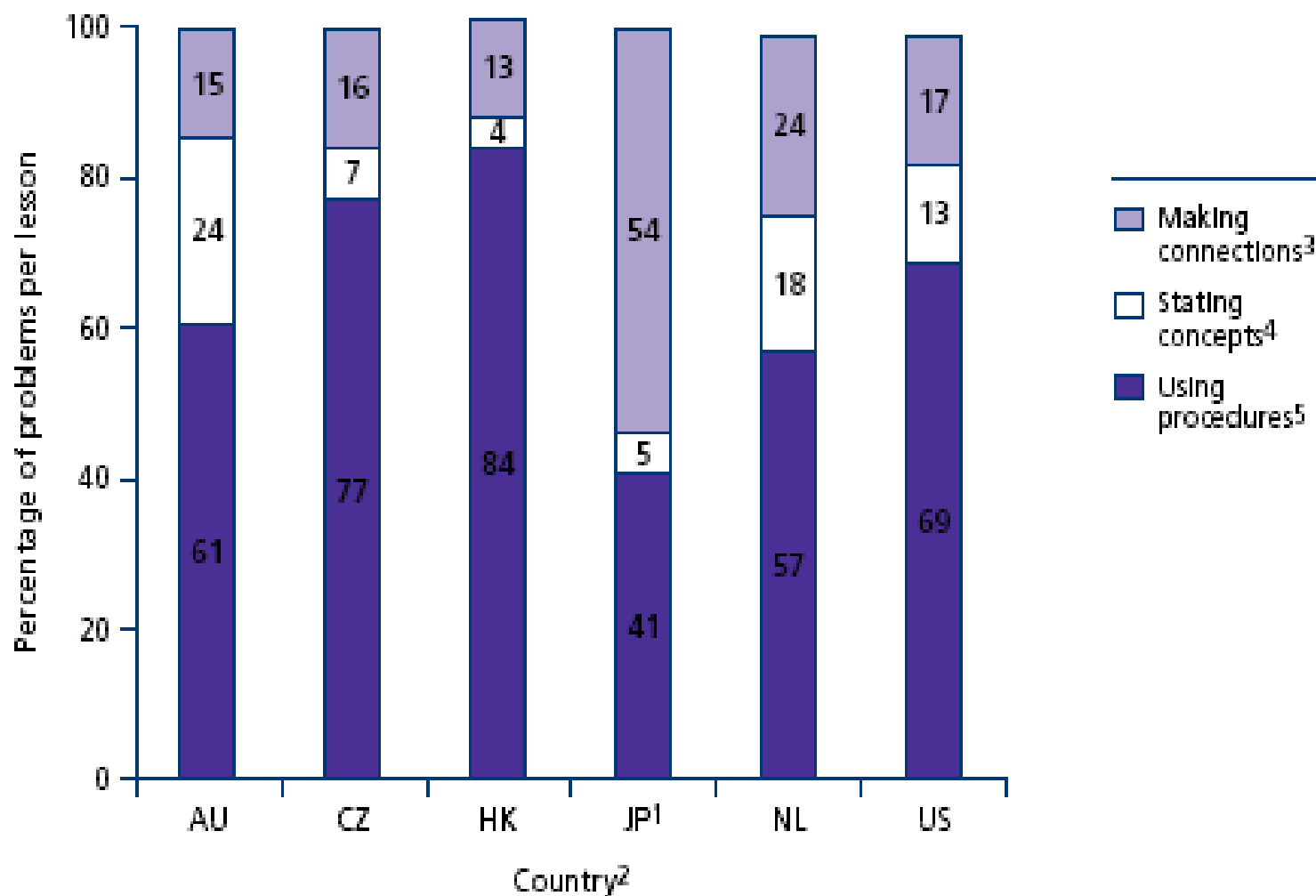




# Australia in comparison

- Many strong features of Australian lessons
  - especially teachers (overwhelming strongest point from our survey)
- Shallow Teaching Syndrome – in comparison Australian Year 8 lessons exhibited:
  - A very high percentage of problems that were **very close repetitions** of previous problems
  - A very high percentage of problems that were of **low procedural complexity** (e.g. small number of steps, not bringing different aspects together)
  - General absence of **mathematical reasoning**
  - [Somewhat low on percentage of problems using real life contexts]
- Aspects of absence of reasoning
  - No lessons contained ‘proof’ (even informal)
  - Very few problems requiring students to ‘make mathematical connections’
  - When problems required connections, these were not emphasised in discussing solutions
    - Often give the result only
    - or often focus on the procedures employed

FIGURE 5.8. Average percentage of problems per eighth-grade mathematics lesson of each problem statement type, by country: 1999



<sup>1</sup>Japanese mathematics data were collected in 1995.

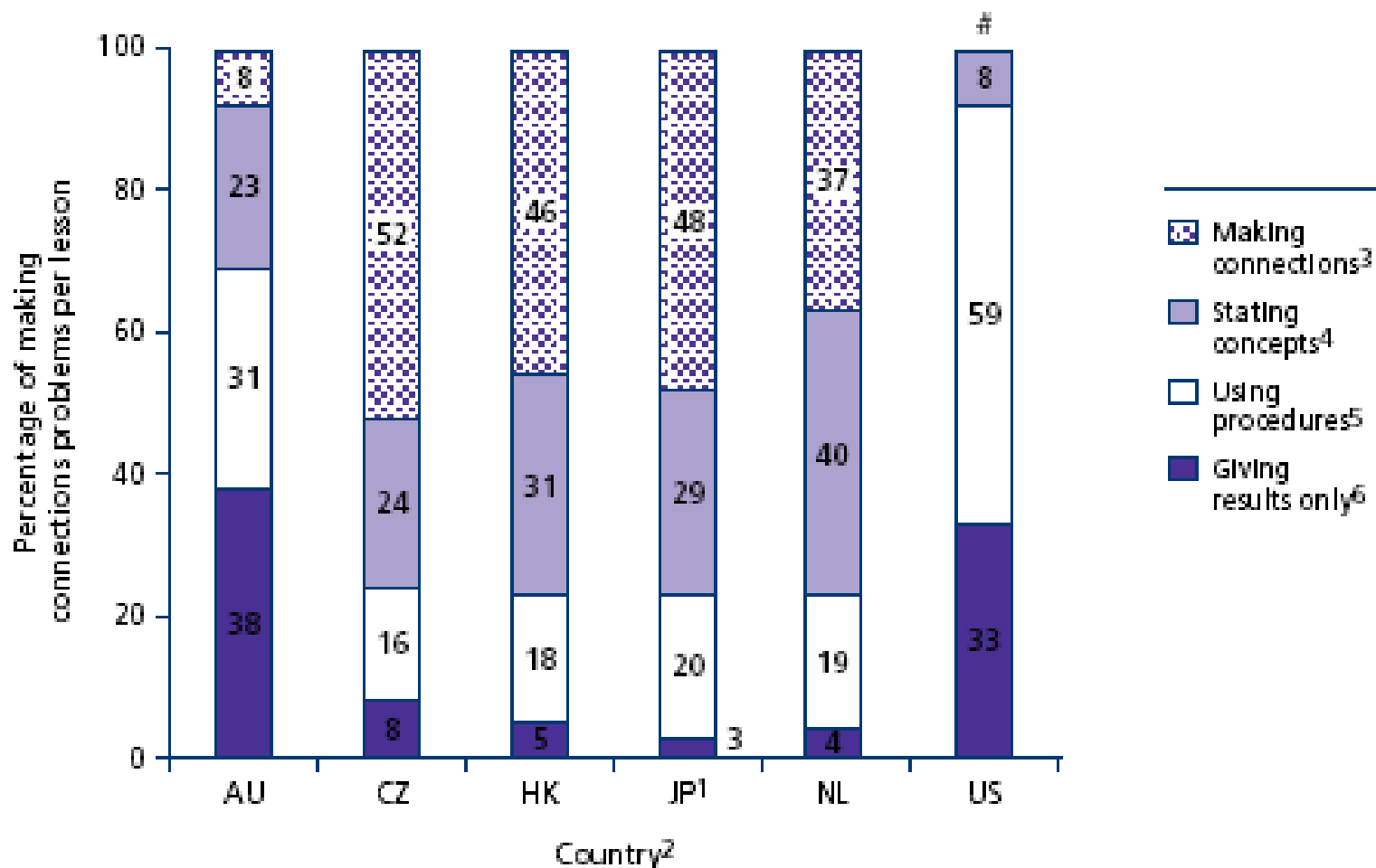
<sup>2</sup>AU=Australia; CZ=Czech Republic; HK=Hong Kong SAR; JP=Japan; NL=Netherlands; and US=United States.

<sup>3</sup>Making connections: JP>AU, CZ, HK, US.

<sup>4</sup>Stating concepts: AU>CZ, HK, JP; NL, US>HK, JP.

<sup>5</sup>Using procedures: CZ>JP, NL; HK>AU, JP, NL, US; US>JP.

**FIGURE 5.12.** Average percentage of making connections problems per eighth-grade mathematics lesson solved by explicitly using processes of each type, by country: 1999



<sup>6</sup>Rounds to zero.

<sup>1</sup>Japanese mathematics data were collected in 1995.

<sup>2</sup>AU=Australia; CZ=Czech Republic; HK=Hong Kong SAR; JP=Japan; NL=Netherlands; and US=United States.

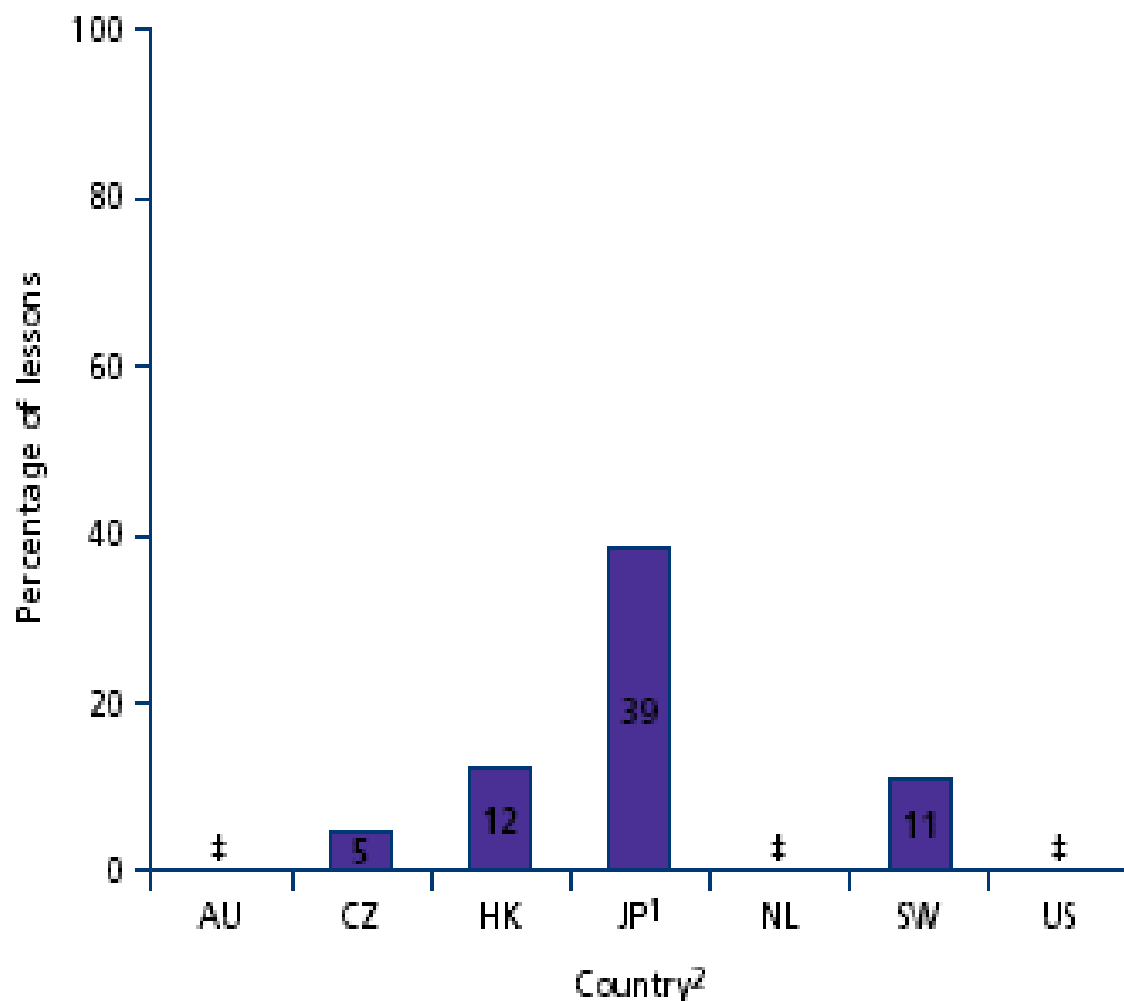
<sup>3</sup>Making connections: CZ, HK, JP, NL>AU, US.

<sup>4</sup>Stating concepts: JP, NL>US.

<sup>5</sup>Using procedures: US>CZ, HK, JP, NL.

<sup>6</sup>Giving results only: AU, US>CZ, HK, JP, NL.

FIGURE 4.4. Percentage of eighth-grade mathematics lessons that contained at least one proof, by country: 1999



<sup>‡</sup>Reporting standards not met. Too few cases to be reported.

<sup>1</sup>Japanese mathematics data were collected in 1995.

<sup>2</sup>AU—Australia; CZ—Czech Republic; HK—Hong Kong SAR; JP—Japan; NL—Netherlands; SW—Switzerland; and US—United States.

NOTE: JP > CZ, HK, SW. The percentage reported for Japan differs from that reported in Stigler et al. (1999) because the definition for proof was changed for the current study.

# Study of Australian Year 8 textbooks

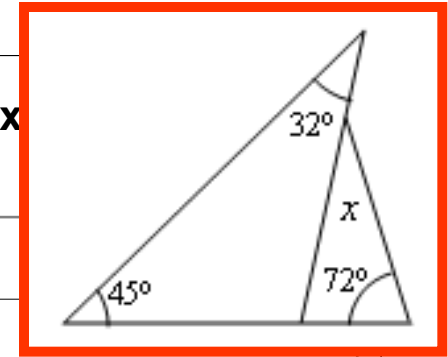
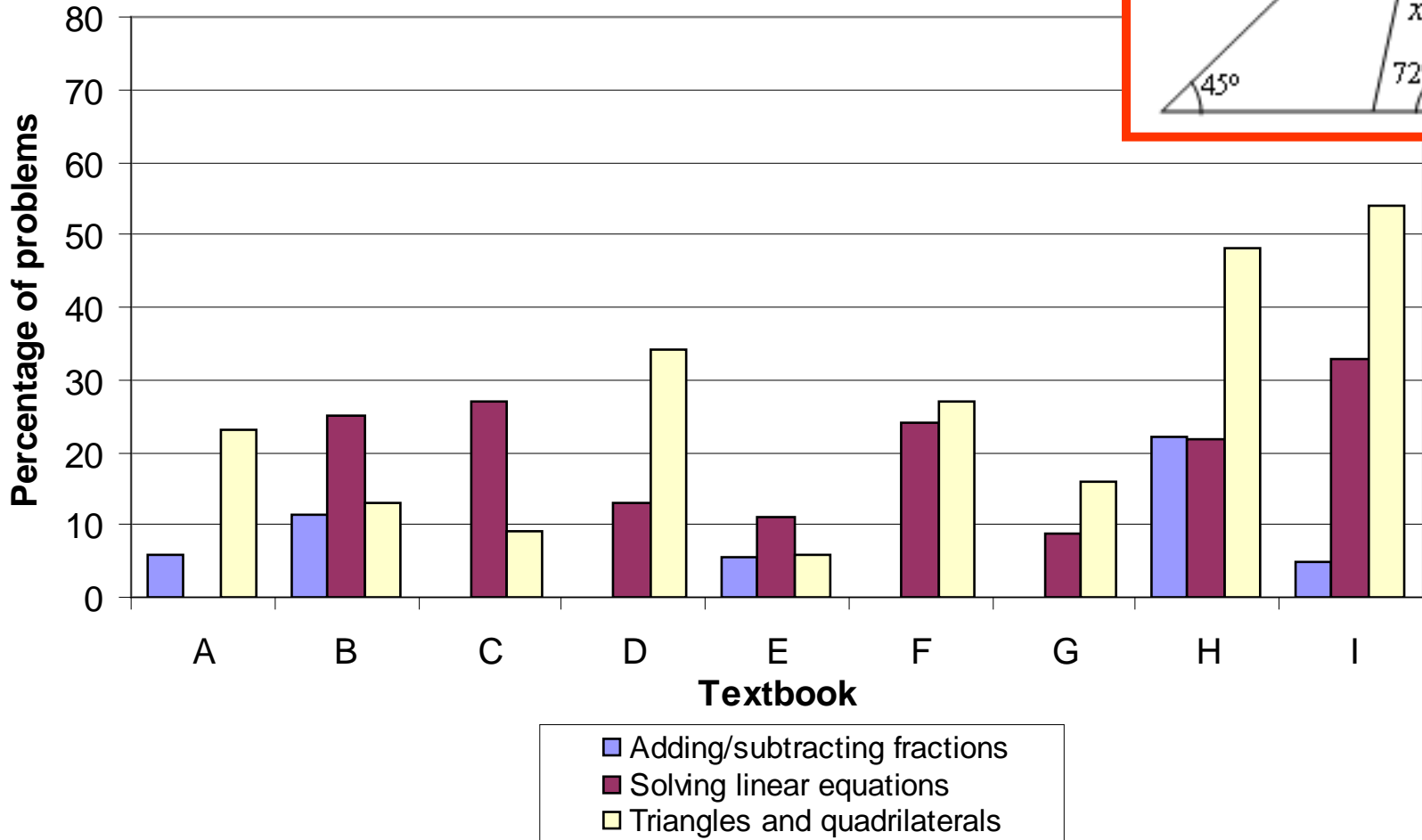


- Chose market-leaders from 4 states and 5 more from one state
  - gives a picture of work presented to many students
- Chose 3 topics
  - Adding and subtracting fractions
  - Solving linear equations
  - Triangles and quadrilaterals
- Classified as in video study (modified from lesson to book) looking at repetition, complexity and requirement for formal or informal proof
- Results:
  - Broadly consistent with Video Study
  - Variation of prevalence between textbooks
  - Tendency for revision of procedures only, and not reasoning
- Work of Jill Vincent and Kaye Stacey – several papers in MERJ

# Applying procedure learned in one context in another context (“application”)

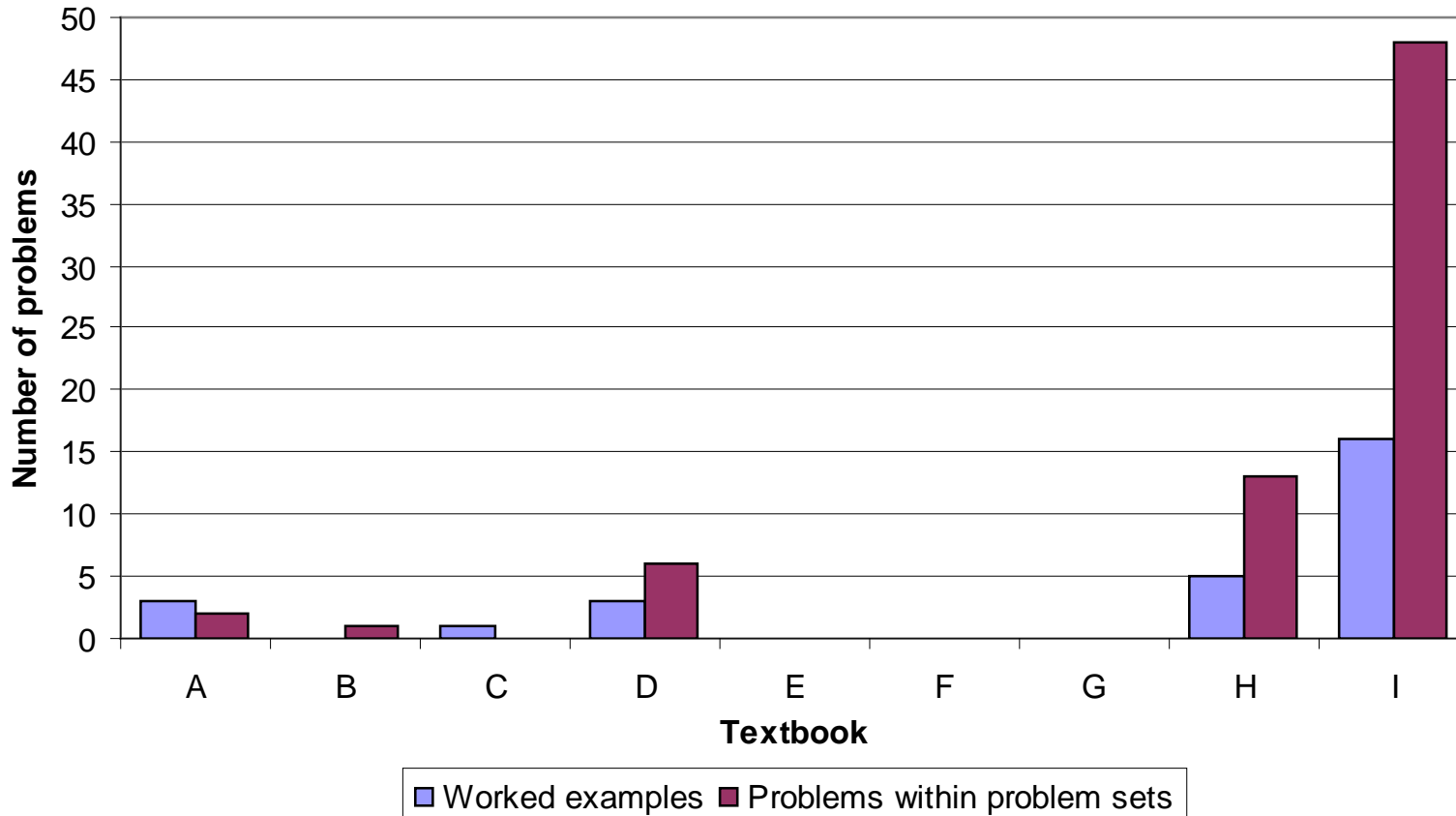


Percentage of problems in sample of eighth-grade textbooks that were applications





## Number of triangle and quadrilateral problems in sample of eighth-grade textbooks that were proofs



Other problems are mainly calculations and ‘naming’ (stating concepts) (e.g, given two angles, find the third; identify type of triangle).  
‘Proof’ here includes informal argument.



# SURVEY OF LEADERS



# Shallow Teaching Syndrome – is it a real and concerning phenomenon?

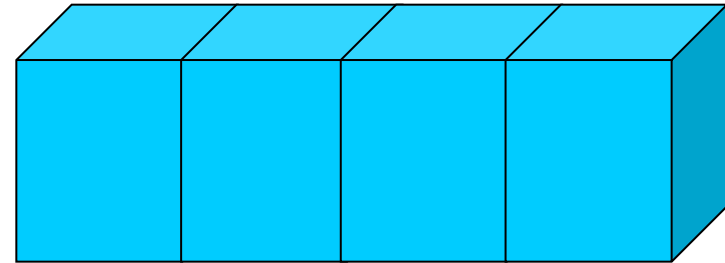


- Is there (ten years later)
  - An undesirably high prevalence of repetition?
  - An undesirably high prevalence of low complexity problems?
  - An undesirable absence of mathematical reasoning?
- Survey of 20 mathematics education leaders – thank you!
  - Participants
    - From departments and systems
    - Textbook authors
    - From mathematics associations
  - Very few differences between these groups
    - association people tend to express more of teachers' view
  - Sometimes difficult to find people with substantial mathematics expertise in departments and systems

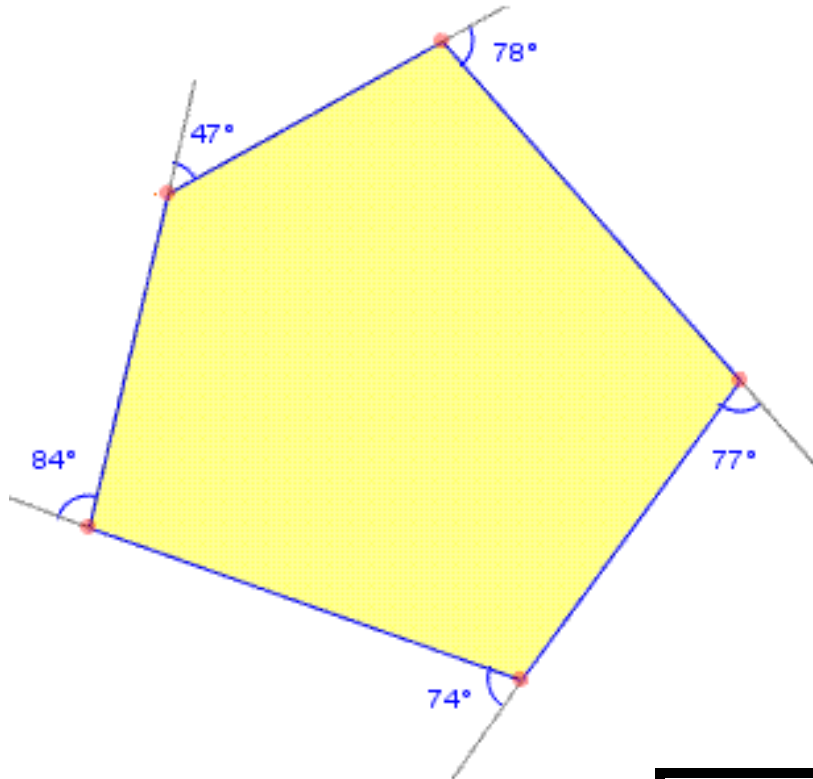
# Examined lessons, textbook extracts, student work, population assessment data



Lessons from  
Several sources

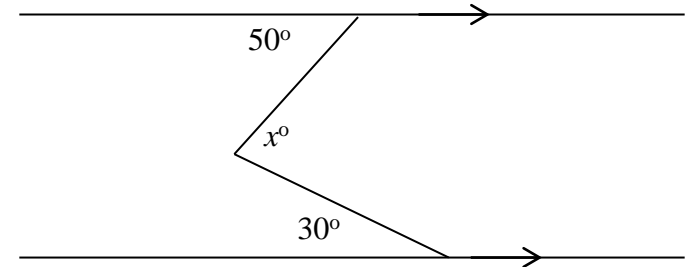


Rule for number of visible faces  
as function of row length



$$77^\circ + 78^\circ + 47^\circ + 84^\circ + 74^\circ = 360^\circ$$

Exercise sets from  
common textbooks



# Shallow Teaching Syndrome – is it a real and concerning phenomenon?



- Is there (ten years later)
  - An undesirably high prevalence of repetition?
  - An undesirably high prevalence of low complexity problems?
  - An undesirable absence of mathematical reasoning?

**Mostly  
qualified yes**

**Repetition and Complexity:  
Questions of balance**

**Mathematical Reasoning:  
Question of 'what'**

**In 10 years,  
computers are  
the main  
change**

**Textbook teaching  
often blamed**

# Is there too much close repetition at Year 8?



- Mixed Response
  - Yes (55%)
  - No (11%)
  - Sometimes (33%)
- Why a lot of close repetition
  - Balancing act with different balance points for different students
    - Confidence and fluency VS Boredom
    - Confidence and fluency VS Lost opportunities for other aspects of mathematics
  - Other reasons
    - Following textbook
    - Behaviour management
    - To maximise results on common forms of assessment
    - Unqualified teachers

# Reasons for perceived high level of repetition in textbooks



- Belief that practice makes perfect – a lot of repetition is the best way (63%)
- Textbook publishers' perception of teachers' preference (31%)
- Easier for textbook authors, writing with few resources (26%)
- Provides security for (unqualified) teachers (31%)
- Provides opportunity for teachers to be selective (31%)
- Catering for different needs in mixed ability classes (21%)

# Procedural complexity



- What are some of the factors that might contribute to Australian lessons containing high levels of problems of low procedural complexity? Do you have reason to think that these factors might operate in Australia more than elsewhere?

The kind of textbook that is used; the nature of teacher content and pedagogical content expertise; and the number of non-maths trained teachers teaching maths. There is currently no system for monitoring teachers' lessons or encouraging teacher dialogue, on-site, and across sites as to how best to teach (refer Lesson Study in Japan).

# Identified roles of problems of low complexity



- Benefits of using low complexity problems
  - Build confidence because accessible for all students
  - Control over introducing new concepts
  - Catering for all abilities
  - Behaviour management
  - Easy for teachers
- Negative consequences
  - Lack of connections being made
  - Limits time for deeper explorations and applications
  - Disengagement of students because of lack of challenge

# Reasons for perceived high prevalence of low complexity problems



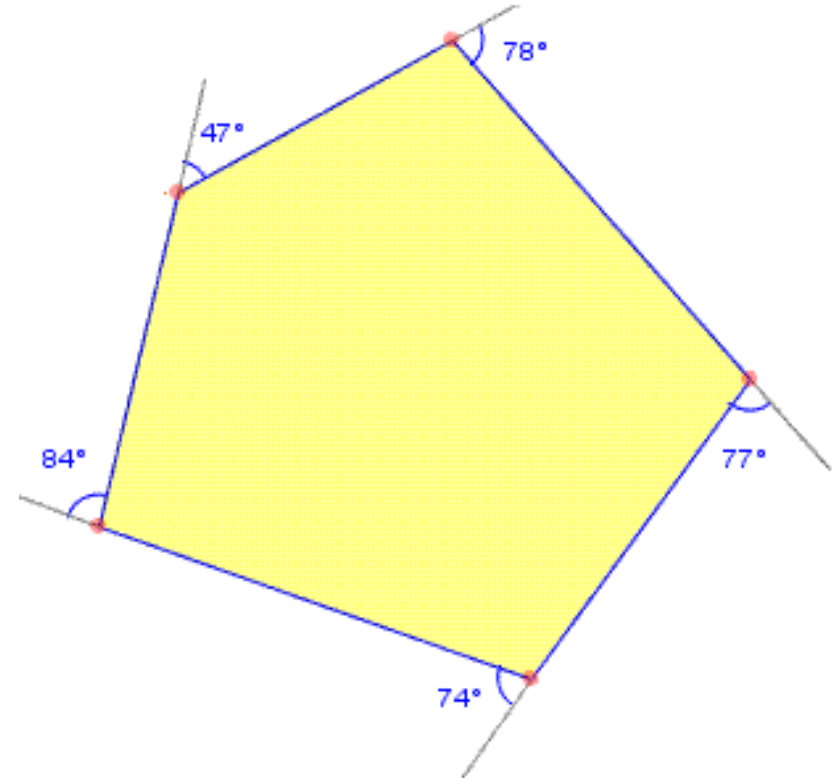
- Under-qualified teachers (50%)
- Behaviour management (33%)
- Use of textbooks (28%)
- Easier for teachers and for students (28%)
- Entrenched teaching methods (22%)
- Low community expectations (22%)
- Time issues (17%)
- Mixed ability classes (11%)
- Perception of a 'good teacher'(11%)



# Varying opinions on nature of appropriate arguments for Year 8



- Role of (possibly informal) proof following a dynamic geometry investigation of sum of exterior angles of a pentagon
  - “I would be happy to students just explore these results”
  - “there are good opportunities for deductive reasoning here”
  - (proof) “not particularly appropriate for many students”
  - “I would not do that”
  - “have students generalise the result to other polygons”
  - “too obvious to prove”



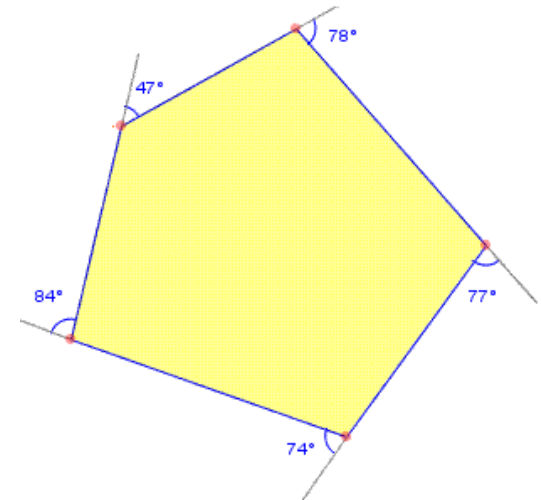
$$77^\circ + 78^\circ + 47^\circ + 84^\circ + 74^\circ = 360^\circ$$

**Sum of exterior angles  
is 360 degrees**

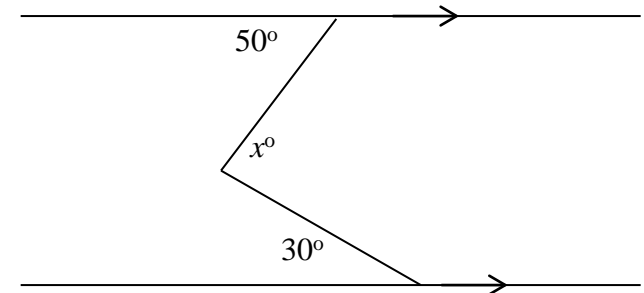


# Great variety of interpretation of WM terms and identification within lessons – missing shared language or shared goals?

- Recognising connections
- Applying problem solving strategies
- Testing conjectures
- Generalising
- Developing arguments
- Following arguments
- Selecting and translating between representations



$$77^\circ + 78^\circ + 47^\circ + 84^\circ + 74^\circ = 360^\circ$$





What reasoning is  
there in textbooks?

# Study of reasoning in 9 Yr 8 textbooks



- Overall aim – examine the mathematical reasoning and proofs presented to Australian students
- Specific Aim
  - examine the reasoning that would be evident in “textbook teaching”
- Examine reasoning in explanations provided
  - look at proofs, verifications, deductions, justifications, explanations etc (Video study ‘PVD’)
  - formal and informal (well, mainly informal, of course!)
- Limitations
  - textbook ↔ student experience
  - not looking at reasoning implied in exercises

# Explanations and proofs



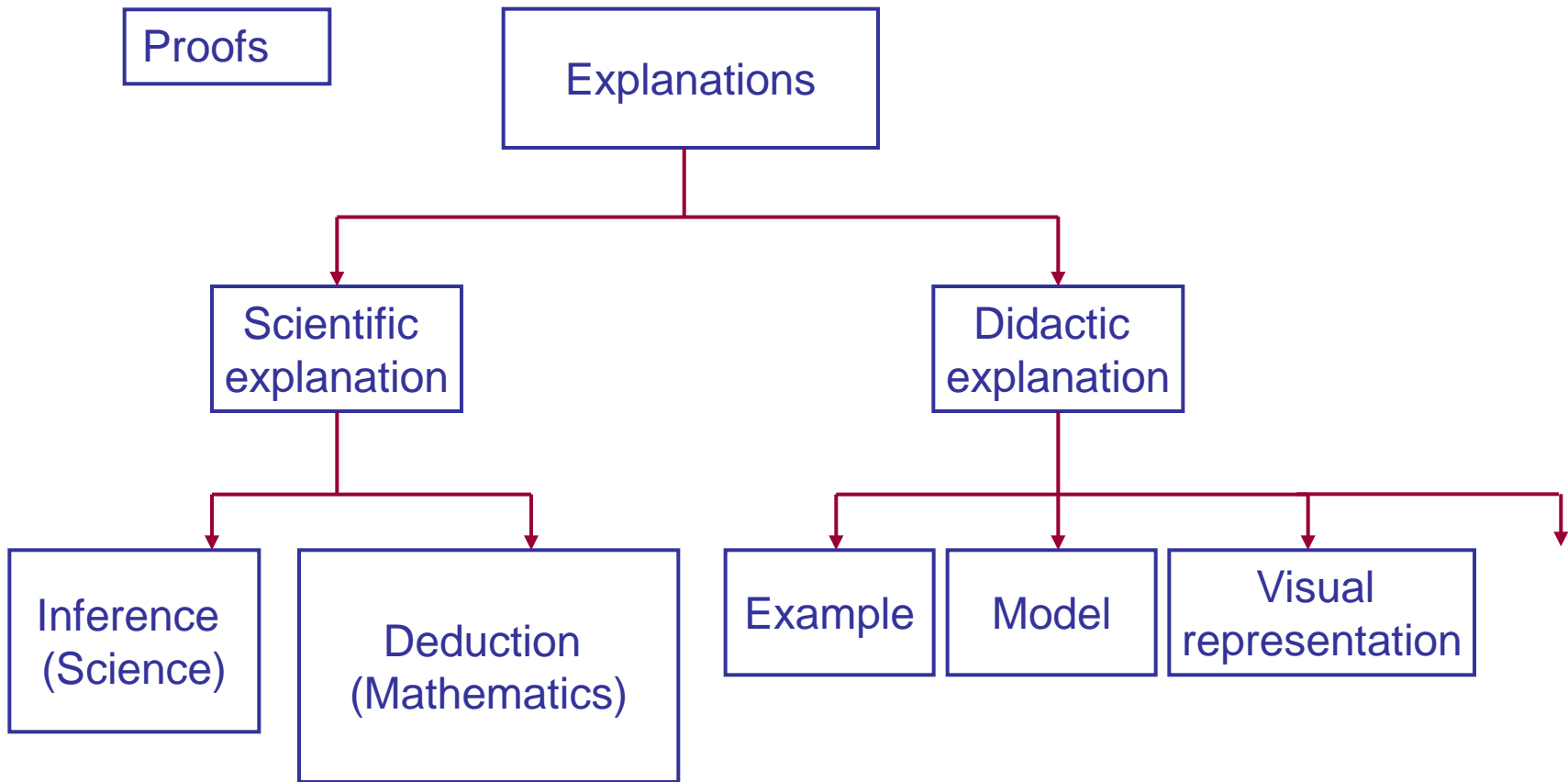
- Explanation is not the same as proof
  - some proofs do ‘explain’ and some do not
  - some explanations are proofs and some are not

“I remember one theorem that I proved, and yet I really couldn’t see why it was true. It worried me for years and years.... I kept worrying about it, and five or six years later I understood why it had to be true. Then I got an entirely different proof. Using quite different techniques, it was quite clear why it had to be true.”

(Michael Atiyah, p. 151 in Mancosu, Jørgensen & Pedersen, 2005)



# Sierpinska (1994)





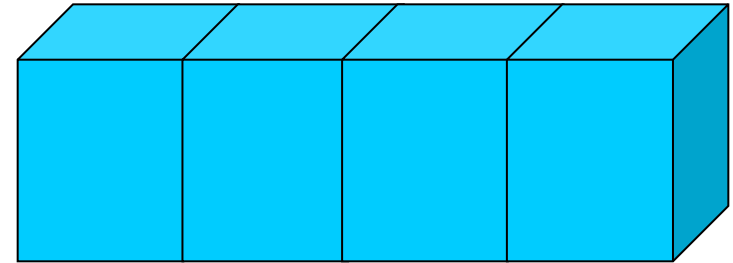
# Method

- Chose market-leaders from 4 states and 5 others from one state
  - gives a picture of explanations presented to many students
- Chose 7 topics where we expected mathematical reasoning was required
  - state differences, so not every topic was in every textbook
- For each topic, we examined all the explanations, justifications and reasoning presented explicitly
  - did not examine implicit reasoning in worked examples, or when solving exercises
  - 53 treatments of 7 topics with 69 distinct explanations
- Each explanation was examined very carefully to identify the nature of the reasoning that supported the critical steps of the argument
- From these examples, a list of the modes of reasoning was created
  - compared to other schemes (e.g. Harel and Sowder; Sierpinksa; Blum & Kirsch)
  - Stacey & Vincent (2009) *Educational Studies in Mathematics*

# Results: Identified 7 modes of reasoning



- deduction using a general case
- deduction using a specific case
- deduction using a model
- concordance of a rule with a model
- experimental demonstration
- qualitative analogy
- appeal to authority



Rule for number of visible faces  
as function of row length – may be  
Deduction using general case  
Deduction using specific case  
Experimental demonstration

**Some explanations are  
easy to categorise;  
some depend on  
interpretation of  
writer's intention**



# Division of fractions

Notes agreement with rule; does not deduce the rule (but it could)

ask the question: *How many twos are there in six?*

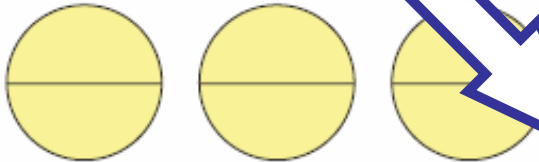


The answer is 3, so  $6 \div 2 = 3$ .

Model 1: Quotition

We can divide fractions in the same way.

For example,  $3 \div \frac{1}{2}$  may be interpreted as *how many halves are there in 3?*



The answer is clearly 6,

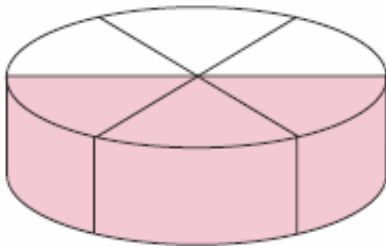
$$\text{so } 3 \div \frac{1}{2} = 6$$

But  $3 \times 2 = 6$  also,

Model 2: Partition

which suggests that dividing by  $\frac{1}{2}$  is equivalent to multiplying by its *reciprocal*, 2.

A further example:



Consider dividing half a cheese equally between 3 people.

Each person would get  $\frac{1}{6}$  of the whole,

$$\text{so } \frac{1}{2} \div 3 = \frac{1}{6}$$

But  $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$  also.



This also suggests that dividing by a number is equivalent to multiplying by its reciprocal.



A working spreadsheet



Multiplication of integers		
Enter the first number here		-11
Enter the second number here	×	-20
This is your answer		220

**Textbook Y: Experimental demonstration OR appeal to authority**

## Multiplication of integers

Consider the patterns in the following multiplication tables.

$$3 \times 3 = 9$$

$$3 \times 2 = 6$$

$$3 \times 1 = 3$$

$$3 \times 0 = 0$$

$$3 \times -1 = -3$$

$$3 \times -2 = -6$$

$$3 \times -3 = -9$$

Result  
decreases  
by 3 each  
step



positive  $\times$  positive = positive  
positive  $\times$  negative = negative

$$-3 \times 3 = -9$$

$$-3 \times 2 = -6$$

$$-3 \times 1 = -3$$

$$-3 \times 0 = 0$$

$$-3 \times -1 = 3$$

$$-3 \times -2 = 6$$

$$-3 \times -3 = 9$$

Result  
increases  
by 3 each  
step



negative  $\times$  positive = negative  
negative  $\times$  negative = positive

*When multiplying two integers with the same sign the answer is positive*

*When multiplying two integers with different signs the answer is negative.*

**Textbook Y: Deduction using a specific case**



# Movies

The movie star can move either *east* (positive direction) or *west* (negative direction). The film can be run either *forwards* (positive) or *backwards* (negative). Fill in the chart showing the person's resulting direction of movement on the screen.

	Direction of the person's movement	Direction that the film is run	Resulting direction of the person's movement on the screen
1.	east (+)	forwards (+)	
2.	west (-)	forwards (+)	
3.	east (+)	backwards (-)	
4.	west (-)	backwards (-)	



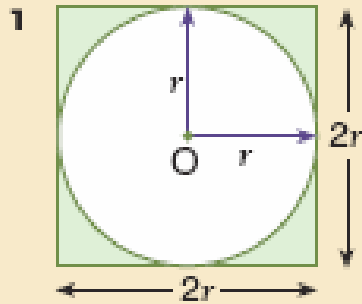
**Textbook Y:  
Qualitative  
Analogy**  
(reason 3)

(but analogical reasoning applies much more widely e.g. using a model)

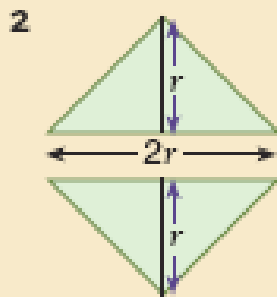
Not a model of multiplication – only signs being “multiplied” e.g.  $+ \times - = -$

(It could be made into a model using velocity of movie star and velocity of film to give apparent velocity of star on film)

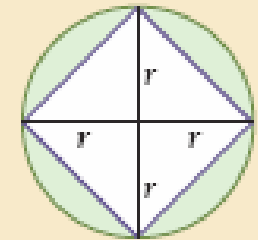
## Investigation 15:04 | The area of a circle



- a This circle with a radius of  $r$  units has been drawn inside a square. Can you see that the sides of the square must be  $2r$  units long? What is the area of the square?
- b Which is larger, the area of the square or the area of the circle?



- a Inside this circle with a radius of  $r$  units, a square has been drawn. The area of the square will be the same as two triangles that each have a base of  $2r$  units and a height of  $r$  units. What is the area of the square?
- b Which is larger, the area of the square or the area of the circle?



From Questions 1 and 2 you should be able to see that the area of a circle lies between  $4r^2$  and  $2r^2$ , ie

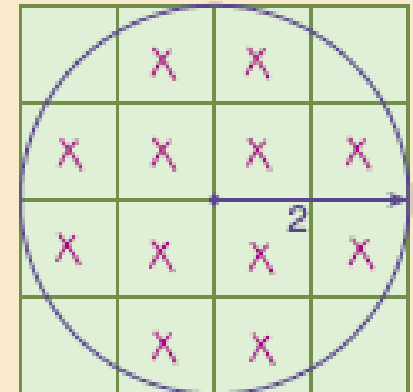
$$2r^2 < \text{area of a circle} < 4r^2$$

So a reasonable approximation for the area of a circle might be

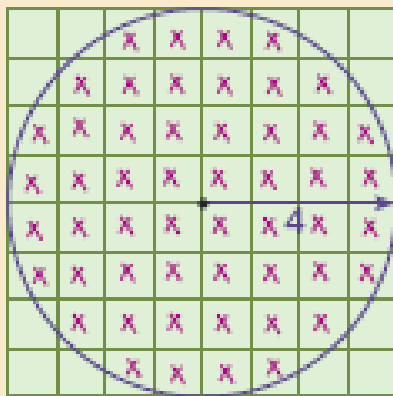
$$A \doteq 3r^2 \text{ square units.}$$

- 3 To gain an approximate value for the area inside a circle, we could 'count squares', including only those for which more than half of the square lies inside the circle.

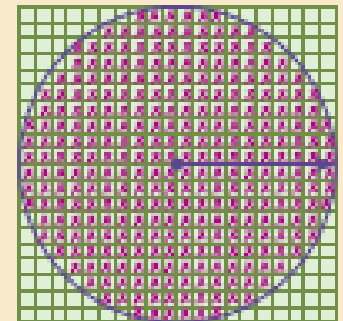
This circle has a radius of 2 units. If we count the squares marked, what is an approximate value for the area of this circle? (How does this compare with the approximate formula  $A \doteq 3r^2$ ?)



- 4 Of course the method in Question 3 can be made more accurate by using smaller square units.



- a Carefully count the squares marked in the circle on the left. What is the approximate area of this circle in square units? How does this compare with  $A \doteq 3r^2$ ?
- b The number of marked squares in the circle on the right, with a radius of 10 units, is 316. If  $A \doteq 3r^2$ , what is the value of  $A$  when  $r = 10$ ?



**Textbook Z: Experimental demonstration**

5

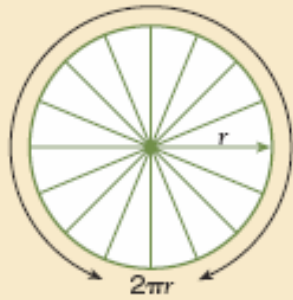


Figure 1

This way of investigating the area of a circle is to slice it into sectors (Figure 1) and then arrange them as in Figure 2.

Figure 2

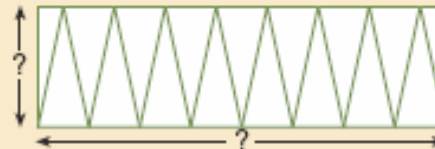
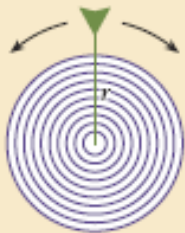


Figure 3

By taking half of the sector on one end and placing it on the other end, we obtain a figure that looks very much like a rectangle, as in Figure 3.

- Now the length of this rectangle would be half the circumference of the circle. What is this length?
- What would be the breadth of the rectangle?
- Since the area of a rectangle is length  $\times$  breadth, what would be the area of this rectangle (which, of course, would be the same as the area of the circle)?

6



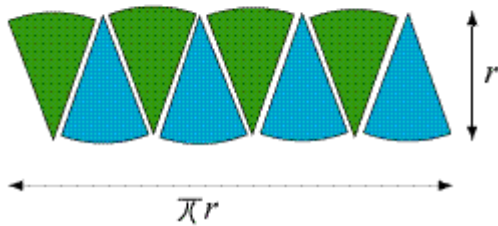
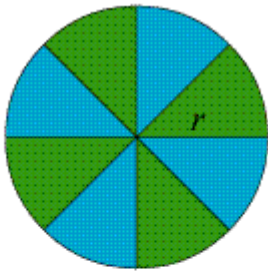
Imagine the area inside a circle to be a series of rubber rings. If the rings were cut along a radius and allowed to fold out flat, the layers would form a triangle.



- What is the formula for the area of a triangle?
- The height of the triangle would be  $r$ , the radius of the circle. What would be the length of the base of the triangle?
- What then would be the area of the triangle?
- What would be the area of the circle?

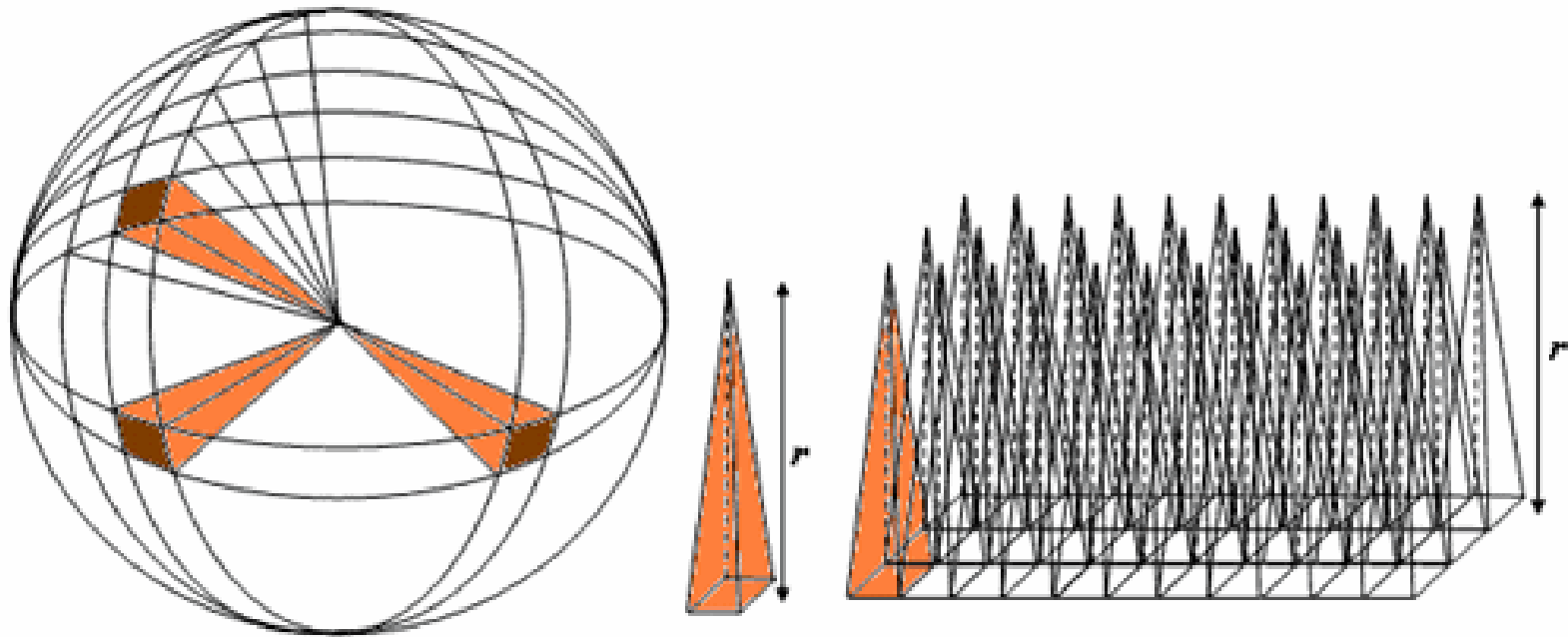


**Textbook Z:  
Deduction using a  
general case  
(guided discovery)**



<http://www.education.vic.gov.au/studentlearning/teachingresources/maths/mathscontinuum/mcd/m47505p.htm>

# Volume of a sphere (Mathematics Developmental Continuum)



<http://www.education.vic.gov.au/studentlearning/teachingresources/maths/mathscontinuum/mcd/M47505sp.htm>





# Observations on explanations

- Very little “rules without reasons” (but certainly some)
  - not what might be expected of “textbook teaching”
- Nearly all (deductive) explanations were ‘correct’ but incomplete, omitting
  - basic reasoning and linking commentary
  - difficult cases and subtle points.
- Explanations brief and unlikely to stand alone
  - students must rely on teachers to elaborate
  - teachers need to know how to do this elaboration (MPCK)
- Purpose appeared to be to derive the rule in preparation for practice exercises
  - little use of explanatory model etc as a thinking tool
  - revision chapters tend to focus on procedures, not reasons

← **Not necessarily bad!**



Multiplying two numbers with like signs gives a positive answer.  
Multiplying two numbers with unlike signs gives a negative answer.  
These statements can be summarised as:

$$+ \times + = +$$

$$- \times + = -$$

$$+ \times - = -$$

$$- \times - = +$$

### **worked example 8**

Calculate:

(a)  $-5 \times -7$

(b)  $-6 \times +9$

#### **Steps**

- (a) 1. Determine the sign of the answer.  
2. Perform the multiplication.  
3. Write the question and answer together, putting in place the correct sign for the answer.  
Remember, if the final answer is + you can leave out the sign.
- (b) 1. Determine the sign of the answer.  
2. Perform the multiplication.  
3. Write the question and answer together, putting in place the correct sign for the answer.

#### **Solutions**

(a)  $- \times - = +$   
 $5 \times 7 = 35$   
 $-5 \times -7 = +35$

(b)  $- \times + = -$   
 $6 \times 9 = 54$   
 $-6 \times +9 = -54$

**After finding the rule, the reasoning is usually not revisited**



# Conclusions about reasoning

- Textbooks used a variety of modes of reasoning
- Variety is between topics, as well as between textbooks
- Four modes of reasoning are unacceptable from a mathematical point of view (to varying degrees):
  - qualitative analogy
  - appeal to authority
  - experimental demonstration
  - concordance of a rule with a model
- Mathematically unacceptable reasoning may or may not be useful pedagogically
  - Do students understand the differences – mathematical reasoning they should learn OR local pedagogical purpose (e.g., to help remember).
  - Are students presented with acceptable (appropriate) reasoning?



# Lesson study video clips Making reasoning more than a ritual before the rule



# APEC Lesson study

- Asia-Pacific Economic Cooperation (APEC) EDNET project on *Classroom Innovation through Lesson Study*
- Organisers: Drs Masami Isoda, Shizumi Shimizu, Maitree Inprasitha, Suladda Loipha, Alan Ginsburg
- Website with many lessons to download from countries around Pacific Rim

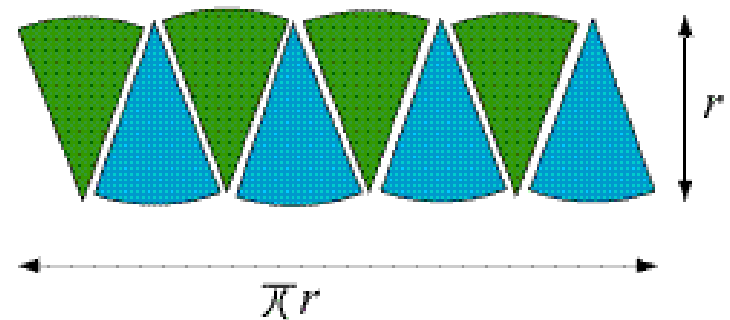
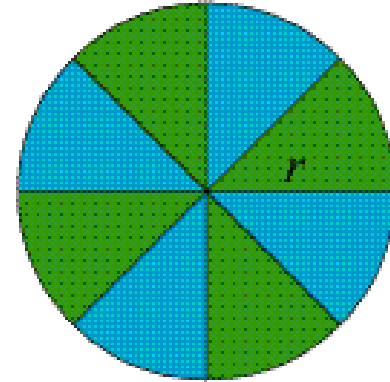
[http://hrd.apec.org/index.php/Area\\_of\\_the\\_Circle\\_Grade\\_5\\_\(Japan\)](http://hrd.apec.org/index.php/Area_of_the_Circle_Grade_5_(Japan))

# Yasuhiro Hosomizu 5<sup>th</sup> grade

## APEC Lesson study video Dec 2006



- Area of circle – sector rearrangement method had been discussed in previous lesson
- Focus of lesson – to find formula for area of circle by rearranging segments of circles into (approximate) shapes with known areas
  - May use triangles, parallelograms, rectangles, trapezoids because these are the shapes with known area formulas



# Extract from lesson plan (from APEC wiki)



deriving the following formula, Area of circle =  $Radius \times Radius \times \pi$  |

**(2) Deriving the formula for finding the area of a circle through activities such as expressing and interpreting ideas by using mathematical expressions**

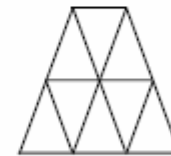
In this lesson, students will have the opportunity to find the area of a circle by rearranging the sectors. These sectors are made from a circle by segmenting it into 8 congruent parts, so that the previously learned formula for finding the area can be used. Then, the students will derive the formula for finding the area of the circle from them. Because manipulating a mathematical expression is very useful during this process, I will emphasize the activities of expressing and interpreting mathematical expressions.



$\frac{1}{8}$  of circumference  $\times$  (radius  $\times 2$ )  $\div 2 \times 4$

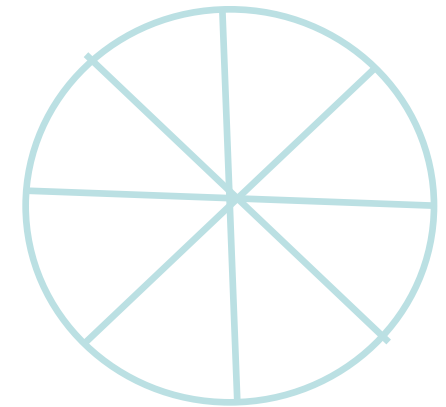
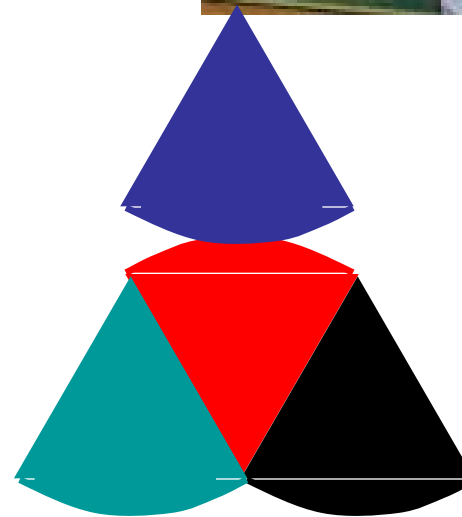
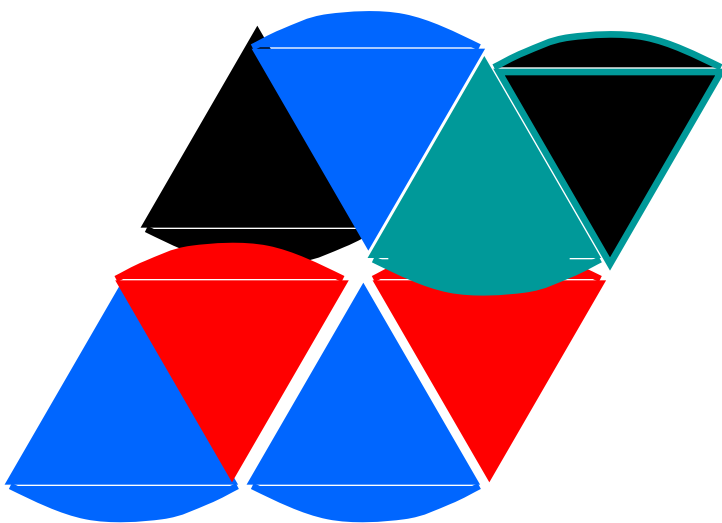
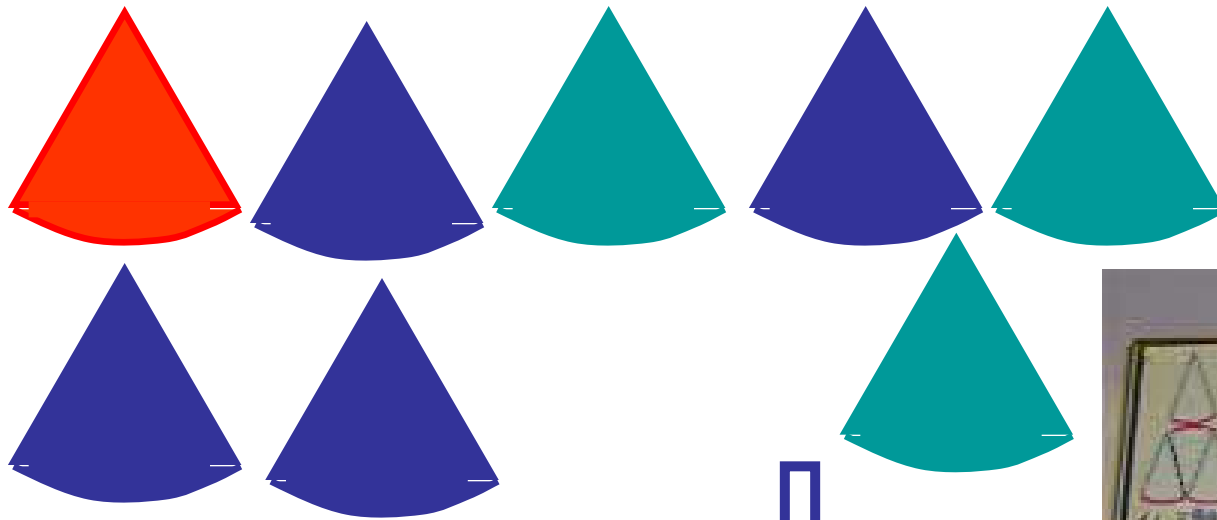


circumference )  $\times$  (radius  $\times 2$ )  $\div 2$



( $\frac{1}{8}$  of circumference +  $\frac{3}{8}$  of

$\frac{1}{4}$  of circumference  $\times$  (radius  $\times 4$ )  $\div 2$





36.45 – 39.45 mins

Area = base x height

= diameter x circumference  $\div$  4

= 20 x circumference  $\div$  4

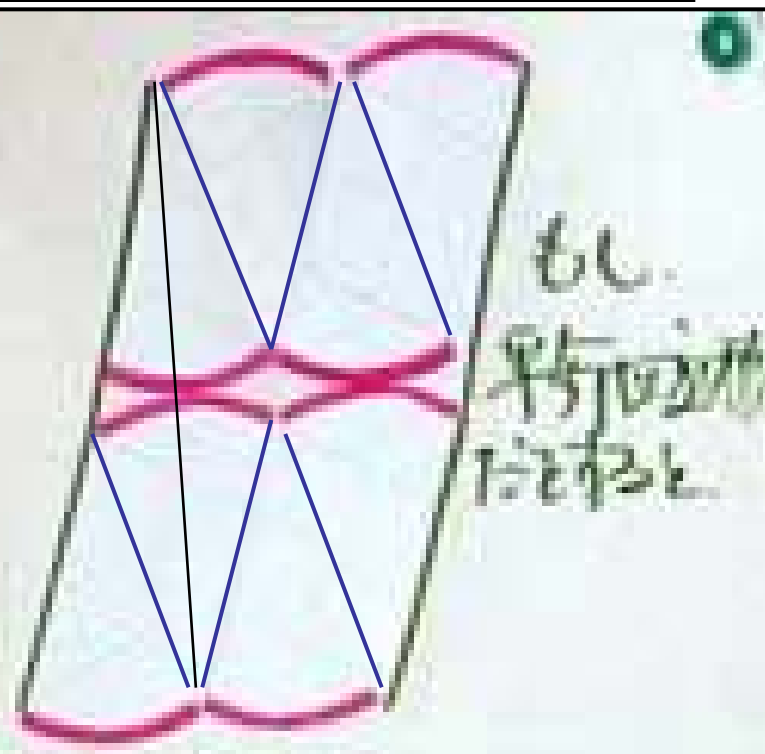
= 5 x circumference

Alternatives:

Diameter  $\div$  4 x circumference

Radius  $\div$  2 x circumference

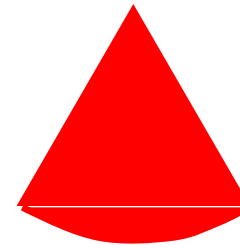
Radius x radius x pi



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- Formulas from parallelogram
  - Half of circumference x radius
  - Radius x pi x radius
  - Diameter x pi ÷ 2 x radius
  - pi x diameter ÷ 2 x radius
  - pi x (radius x 2) ÷ 2 x radius
  - pi x radius x radius (video at about 20 mins)



*Also  
alternate  
between  
pi and  
3.14*

# Observations

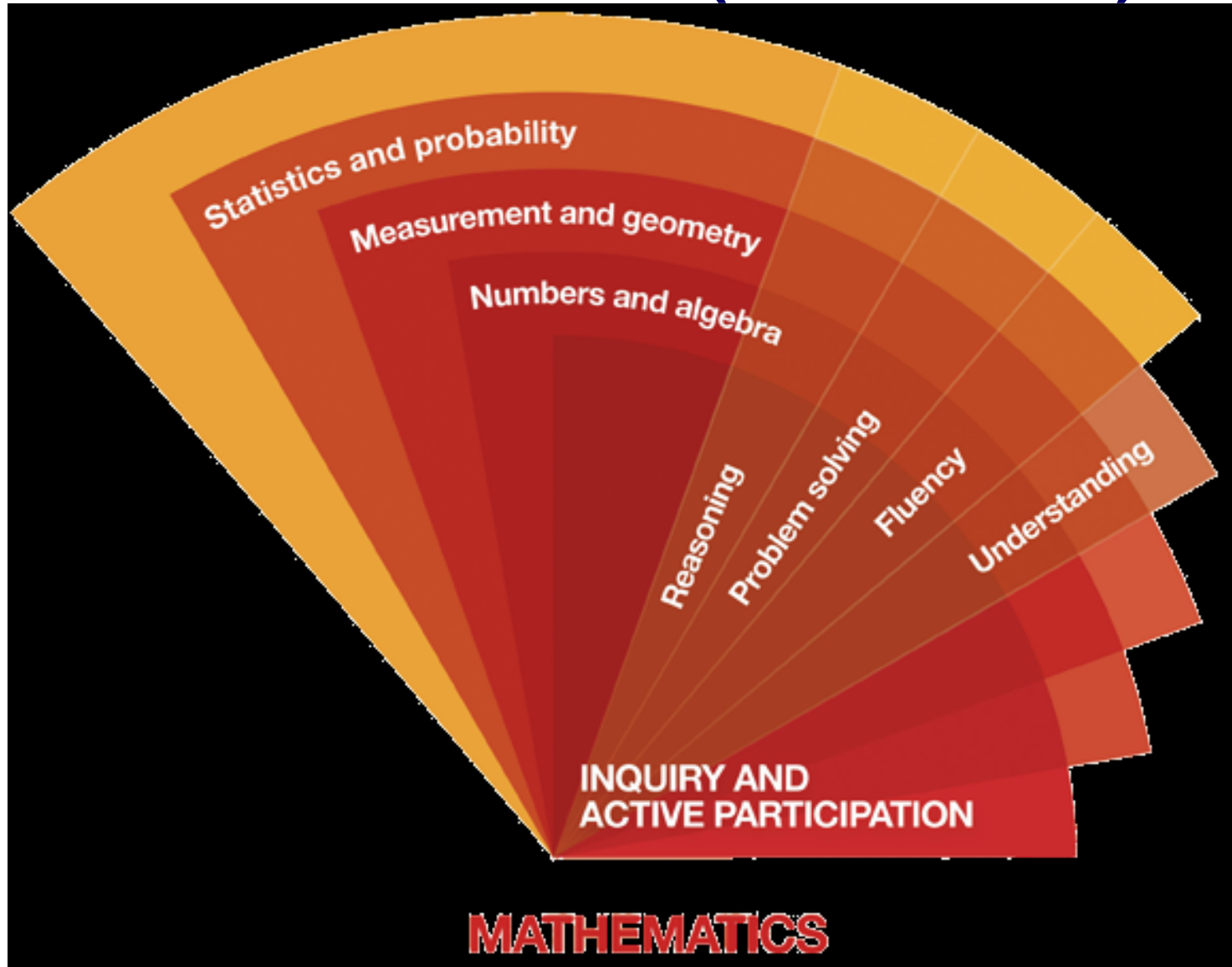


- Some very well prepared students, with creative ideas
- Value in repeating the reasoning and argument with different shapes – even these excellent students need to revisit the ideas
- Even expert teacher finds it hard to use children’s explanations productively
  - Shift by one boy between the general and particular (at 36 mins) makes it very hard for other students to follow reasoning
- Often even harder to use wrong reasoning productively
- Focus on deep mathematical principles in lesson – e.g. using the area formulas that you know
- Secondary aim of expressions: Difficulty and multiple steps in moving between expressions



# Reasoning in the Australian Curriculum

# Australian Curriculum (2010 March)



# Australian Curriculum Proficiency Strands



- Understanding
- Fluency
- Problem Solving
  - Students develop the ability to make choices, interpret, formulate, model, and investigate problem situations, and communicate solutions effectively
- Reasoning
  - Students develop increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising
- Also “general capabilities” across subjects which include “thinking skills” and “creativity” (p. 5)

# Mathematics content and process – concerns



- In the survey of leaders, all were concerned (although details differ) and all recommend
  - Better resources
  - Teacher education
  - Therefore important that Australian Curriculum provides sound advice
- Different nature of four proficiency strands means they need different treatment
  - Understanding and fluency – inherent part of learning content well
  - Problem solving and reasoning – more than this
    - Important outcomes of learning, independent of content
    - Part of the fabric of any real mathematics lesson
    - Also contributing to learning content
- Dilemma of separation from content VS integration with content
  - in class and in curriculum specification
  - Need to identify relevant goals for broad age groups

**Everywhere and nowhere**

# Reasoning and PS are not just “forms” of classroom interaction (e.g. discussion)



- Reasoning is not evident in the form of classroom interaction, but in the substance
- External evidence of reasoning
  - Classroom discussion
    - Find reasons and arguments
    - Compare reasons and arguments
    - Analyse reasons and arguments
  - Writing arguments
    - Consolidate reasoning
    - Check reasoning
- Reasoning happens when students work by themselves too!





# To encourage reasoning:

- Some indication of how reasoning develops, and characteristics of different stages
  - That both empirical and deductive argument are present from the early years of school
  - Some indication of developing complexity (e.g. length of argument)
  - Algebra provides a language for generality, but this does not mean that students do not make general arguments before this (“deduction using a specific case”)
- Consistent examples of appropriate mathematical reasoning at each level
  - Example: the role of definitions in mathematics
  - Give examples which reinforce reasoning, not just practice rules



# Argument from definition

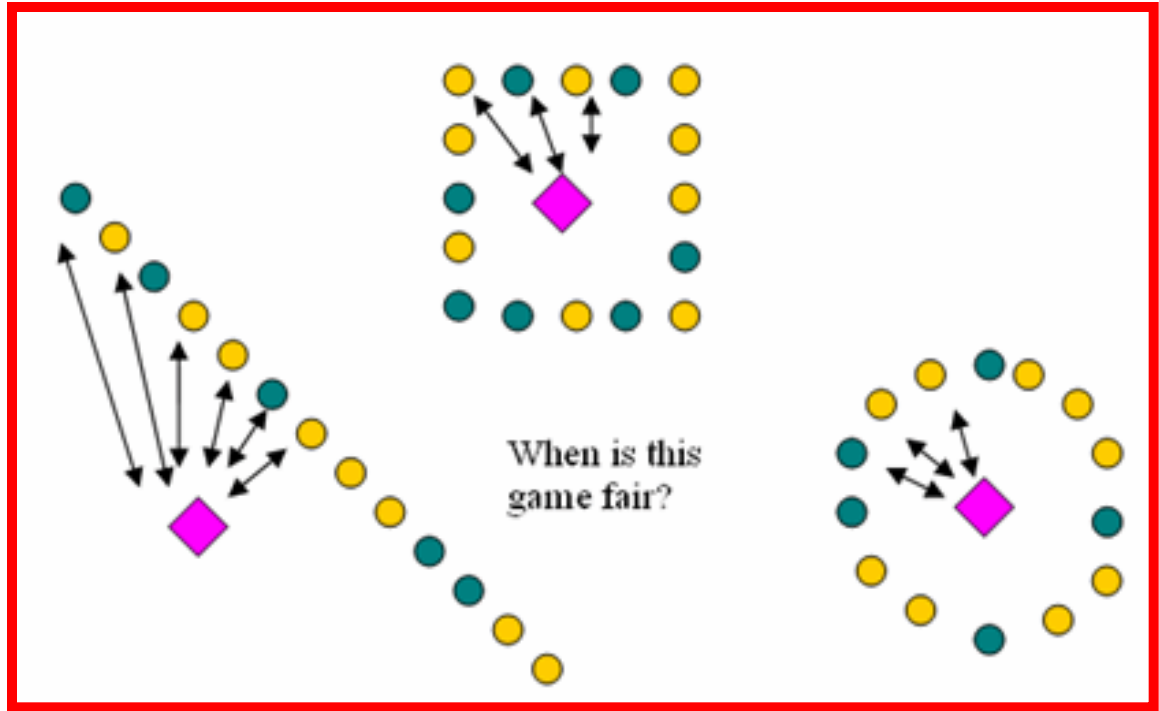
- A very strong feature in mathematical reasoning
- Must be strengthened in curriculum document by attention to detail



# Australian Curriculum - Circles

- Year 1 (p. 10) describe shapes
  - “saying circles are round”
- Year 2 (p.14)
  - “sorting circles, triangles and rectangles and saying the grouping is based on the number of straight sides”
  - Year 5 (p.33) “noting similarities such as all quadrilaterals have 4 straight sides”
- Year 7 ( p.46)
  - [using pairs of compasses for construction]
- Year 8 (p 51 et seq)
  - Considerable circle geometry

Australian Curriculum Yr 1 (p.10)  
"saying that circles are round"





# even + even = even

Definition of even (and hence odd) numbers

- Multiple of 2
- Last digit is 2, 4, 6, 8, 0
- Is made up from pairs

Empirical (experimental demonstration) – look at examples

$$1 + 1 = 2 \quad 2 + 4 = 6 \quad 3 + 3 = 6$$

$$2 + 6 = 8 \quad 13 + 5 = 18$$

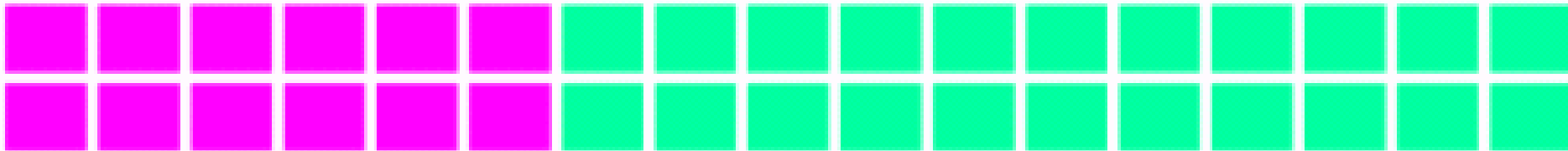
$$12 + 4 = 16$$

$$14 + 5 = 19$$

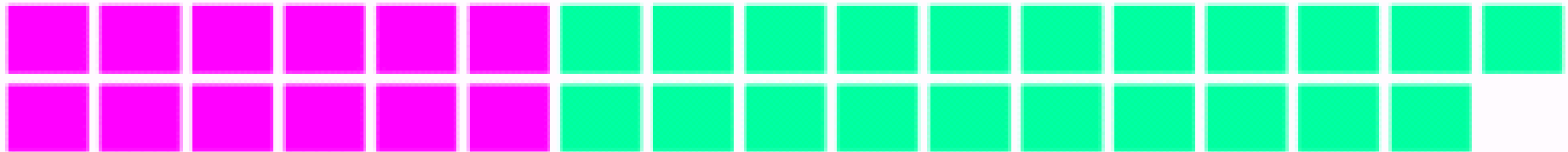


# Argument available to young students – but it depends on definition

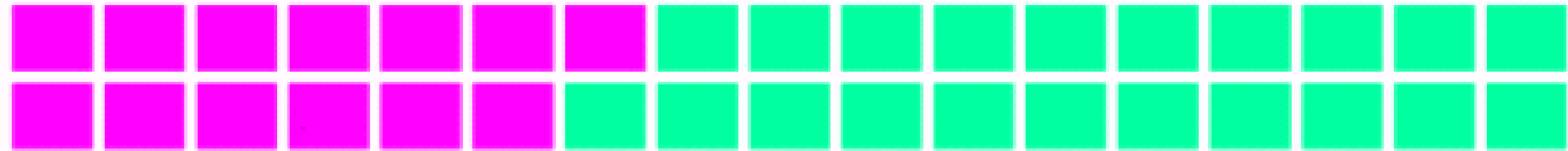
even + even = even



even + odd = odd



odd + odd = even



Note “seeing the general in the particular” (Deduction using a special case)



<http://www.fgworld.co.uk/Schooldays/Mary%20Hines/st%20mikes%20walking%20day%20kids.jpg>



# Explanation?

To prove that the sum of two even numbers is even

Let  $a$  and  $b$  be even numbers

Then

$$a = 2m \text{ for } m \in \mathbb{Z}$$

$$b = 2n \text{ for } n \in \mathbb{Z}$$

$$\text{Therefore } a + b = 2m + 2n = 2(m + n)$$

Since  $(m + n) \in \mathbb{Z}$ ,  $a + b$  is an even number

Definition of even (and hence odd) numbers

- Multiple of 2
- Last digit is 2, 4, 6, 8, 0
- Is made up from pairs

T  
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m





# Summary

- Understanding and fluency strands
  - These relate to the quality of learning content goals
- Problem solving and reasoning strands
  - These relate to the quality of learning content goals
  - Also more elusive but fundamental goals
- Repetition and low complexity : Out of balance!
  - “justified” by the fluency goal
  - but there are many other reasons
- Reasoning
  - Many types of reasoning in good math’l didactic explanation
  - Highlight the mathematical ones, use others knowingly
  - Review the reasoning – don’t make it just a starting ritual
- Draft Australian curriculum – aim to make maths teaching better
  - Clarify goals for reasoning throughout
  - Build towards reasoning in every aspect



# Thank you

**The reported studies are joint work with Dr Jill Vincent**

**ACER Research Conference**

**15 – 17 August, 2010**

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