Standards, what's the difference?: A view from inside the development of the Common Core State Standards in the occasionally United States

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Abstract

Standards sequence as well as express priority. On what basis? Learning trajectories sequence through empirical investigation and theory. The sequence, as far as it goes, has empirical validity, but only some sequences have been developed. Standards, in contrast, must choose what students need to learn as a matter of policy. This article will discuss issues of sequence, focus and coherence in mathematics standards from the perspective of the Common Core State Standards (CCSS) for Mathematics in the United States of America.

Decisions about sequence in standards must balance the pull of three important dimensions of progression: cognitive development, mathematical coherence, and the pragmatics of instructional systems. Standards are written as though students in the class have learned approximately 100 per cent of preceding standards. This is wild fiction in any real classroom. This difference between the genre convention of 'immaculate progression' in standards and the wide distribution of student readiness in real classrooms is a dangerous difference to ignore. Each student arrives at the day's lesson with his or her own mathematical biography, whatever the student learned on their personal trajectory through mathematics. A spectacular diversity of such *personal learning trajectories* (PLoTs) faces the teacher at the beginning of each lesson. There are two related manifolds in play during each lesson: the manifold of PLoTs (personal learning trajectories) in the classroom and the manifold of learning trajectories (LTs) that enable the learning of the mathematics being taught. As real as these trajectories may be, neither is in plain sight. What is in plain sight are standards, tests, textbooks and students.

LTs are too complex and too conditional to serve directly as standards. Still, LTs point the way to optimal learning sequences and warn against hazards that could lead to sequence errors. Teachers and students need time within the lesson and across the unit to pull students from PLoTs along LTs to the SSTs. This requires standards to be within reach.

The types of errors in the way standards might be sequenced are reviewed.

Introduction

> *One sees the difficulty with this standards business. If they are taken too literally, they don't go far enough, unless you make them incredibly detailed. You might give a discussion of a couple of examples, to suggest how the standards should be interpreted in spirit rather than by the letter. But of course, this is a slippery slope.*
>
> Roger Howe, Yale,
> March 15, 2010
> input to common core standards

> *… the "sequence of topics and performances" that is outlined in a body of mathematics standards must also respect what is known about how students learn. As Confrey (2007) points out, developing "sequenced obstacles and challenges for students… absent the insights about meaning that derive from careful study of learning, would be unfortunate and unwise." In recognition of this, the development of these Standards*

began with research-based learning progressions detailing what is known today about how students' mathematical knowledge, skill, and understanding develop over time.

Common Core State Standards, 2010

Sequence, Coherence and Focus in Standards and Learning Trajectories

Learning trajectories sequence levels of cognitive actions and objects through empirical investigation and theory. As result the sequence has empirical validity. However, the question of *what* is being sequenced is a matter of researcher choice, often driven by theoretical considerations related to a trajectory of interest to the *researcher*. Some researchers (Clements and Sarama, 2010 {this report}) suggest these choices include consultation with mathematicians and educators to obtain valid focus. Still, the choice of what mathematics gets research attention is not, in itself, a valid basis for deciding what to teach. Standards, in contrast, begin with choices about what students need to learn as a matter of policy.

Standards, perforce, sequence as well as express priority. On what basis? By design, at least, one hopes. To what extent can and has the design of mathematics standards been informed by research and empirically well founded theories of learning trajectories? This article will contemplate that question for the recently developed Common Core State Standards in mathematics, the closest this nation has ever come to national standards. It is an interesting tale that leads to fundamental, perhaps very productive, questions about standards and trajectories, and their consequences for instruction, curriculum, assessment and the management of instruction.

This article will look at the general issues of sequence, focus and coherence in mathematics standards from the perspective of the Common Core State Standards (CCSS) for Mathematics. I was a member of the small writing team for the CCSS. As such, I was part of the design, deliberation and decision processes, including especially reviewing and making sense of diverse input solicited and unsolicited. Among the solicited input were synthesised 'progressions' from learning progressions researchers.

Grade level vs. development

Standards sequence for grade levels; that is, the granularity of the sequence is year-sized. Standards do not explicitly sequence within grade level, although they are presented in some order that makes more or less sense. Sometimes this order within grade is compelling, thus luring users to over interpret the within grade presentation as teaching sequence.

From the start, we encounter a problematic convention: standards are written as though students have learned everything (100%) in the standards for the preceding grade levels. No one thinks most students have learned 100%, but this genre convention for standards seems a sensible approach to avoiding redundancy and excessive linguistic nuance. But how does this mere genre convention drive the management of instruction? Test construction? Instructional materials and their adoption? Teaching? Expectations and social justice? Ah…the letter or the spirit and the slippery slope.

Cognitive development, mathematical coherence and pedagogic pragmatics

Decisions about sequence in standards must balance the pull of three important dimensions of progression: cognitive development, mathematical coherence, and the pragmatics of instructional systems. The situation differs for elementary, middle and high school grades. In brief: elementary standards can be more determined by research in cognitive development and high school more by the logical development of mathematics. Middle grades must bridge the two, by no means a trivial span.

For example, the Common Core State Standards (CCSS) incorporate a progression for learning the arithmetic of the base ten number system. A logical development mathematically would begin with sums of terms which are products of a single digit number and a power of ten, including rational exponents for decimal fractions. Yet no one thinks this is the way to proceed. Instead, the CCSS for grade 1 ask students to,

2. Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:

- **a.** 10 can be thought of as a bundle of ten ones—called a "ten."
- **b.** The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones. …

The relative weight to give cognitive development vs. mathematical coherence gets more tangled with multiplication, the number line and especially fractions. In third grade, the CCSS introduces two concepts of fractions:

1. Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.

2. Understand a fraction as a number on the number line; represent fractions on a number line diagram.

   a. Represent a fraction $1/b$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $1/b$ and that the endpoint of the part based at 0 locates the number $1/b$ on the number line.

   b. Represent a fraction a/b on a number line diagram by marking off a lengths $1/b$ from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line.

The first concept relies on student understanding of equal partitioning. Jere Confrey (2008) and others have detailed the learning trajectory of children that establishes the attainability of this concept of fraction. Yet by itself, this concept is isolated from broader ideas of number that, for the sake of mathematical coherence, are needed early in the study of fractions. These ideas are established through the second standard that defines a fraction as a number on the number line. This definition has a lot of mathematical power and connects fractions in a simple way to whole numbers and, later, rational numbers including negatives (Wu, H., 2007). Simple looking forward, but mysterious coming from prior knowledge.

The Writing Team of CCSS received wide and persistent input from teachers and mathematics educators that number lines were hard for young students to understand and, as an abstract metric, even harder to use in support of learning other concepts. Third grade, they said, is early for relying on the number line to help students understand fractions. We were warned that as important as number lines are as mathematical objects of study, number lines confused students when used to teach other ideas like operations and fractions. In other words, include the number line as something to learn, but don't rely on it to help students understand that a fraction is a number.

The difference in advice on fractions on the number line was not easy to sort through. In the end, we placed the cognitively sensible understanding first and the mathematical coherence with the number line second. We included both and used both to build understanding and proficiency with comparing and operations with fractions.

Does the number line appear out of the blue in third grade? No. We looked to the research in learning trajectories for measurement and length to see how to build a foundation for number lines as metric objects (Clements, 1999c; Nührenbörger, M., 2001; Nunes, T., Light, P., and Mason, J.H. 1993). The Standards from Asian countries like Singapore and Japan were also helpful in encouraging a deeper and richer development of measurement as a foundation for number and quantity.

Clements and Sarama (2009) emphasize the significance of measurement in connecting geometry and number, and in combining skills with foundational concepts such as conservation, transitivity, equal partitioning, unit, iteration of standard units, accumulation of distance, and origin. By around age 8, children can use a ruler proficiently, create their own units, and estimate irregular lengths by mentally segmenting objects and counting the segments.

The CCSS foundation for the use of the number line with fractions in 3rd grade can be found in the 2nd grade Measurement standards:

Measure and estimate lengths in standard units.

- Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.
- Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.
- Estimate lengths using units of inches, feet, centimeters, and meters.
- Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

Relate addition and subtraction to length.

- Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.
- Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, …, and represent whole-number sums and differences within 100 on a number line diagram.

This work in measurement in 2nd grade is, in turn, supported by 1st grade standards:

- Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. *Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.*

This sequence in the CCSS was guided by the learning trajectory research. This research informed the CCSS regarding essential constituent concepts and skills, appropriate age and sequence. Yet the goal of having number line available for fractions came from the need for mathematical coherence going forward from 3rd grade, rather than from learning trajectory research.

Instructional Systems and Standards

Perhaps the most important consequence of standards is their impact on instruction and instructional systems. This impact is often mediated by high stakes assessments which will be dealt with later. Two crucial instruction issues will be discussed that are too often buried in comforting cushions of unexamined assumptions. The first issue is, how do the structure, properties and behavior of mathematics knowledge interact with instruction?

The second issue arises from the way standards are written, as though students in the middle of grade 5 have learned approximately 100% of what is in the standards for grade k-4 and half of 5. This is never close to true in any real classroom. This difference between the genre convention of "immaculate progression" in standards and the wide distribution of student readiness in real classrooms has important consequences. It means, for one thing, that standards are not a literal portrayal of where students are or can be at a given point in time. And, for me, the negation of 'can' negates 'should'. Standards serve a different purpose. They map stations through which students are lead from wherever they start.

Immaculate progression literalism has contributed to confusion about what "proficient" means as a test result. Most state tests have "proficient" cut scores at 60% or less (with guessing allowed on multiple choice, [usually 4 choices], items that make up close to all of the test). Thus even the distribution of 'proficient' students lacks large chunks of learning of the standards, at least as assessed by the standards based test.

The rough terrain of prior learning where lessons live

The standards based curriculum is a sequence through the calendar: year to year, month to month, day to day. Think of this as a horizontal path of concepts and skills. Such a path can match textbooks and tests, but never the distribution of students in a classroom. Beneath the surface of the standards sequence trajectory (SST) is the underwater terrain of prior knowledge. Each student arrives at the day's lesson with his or her own mathematical biography, whatever the student learned on their personal trajectory through mathematics. A spectacular diversity of such personal learning trajectories (PLoTs) faces the teacher at the beginning of each lesson (Murata, A., & Fuson, K. C., 2006).

The teacher, on the other hand, brings to this diversity an ambition for some mathematics to be learned. The mathematics has a location in yet another trajectory: the logical sequence of ideas which reflects the deductive structure of mathematics (MTs). Thus, there are three related manifolds in play: the PLoTs (personal learning trajectories) in the classroom, the MTs and the learning trajectories (LTs). As real as these trajectories may be, none are in plain sight.

…teaching is like riding a unicycle juggling balls you cannot see or count.

What is in plain sight are standards, tests, textbooks and students. A teacher cannot actually know the students' PLoTs. Nor has research mapped the territory of the standards with LTs.. And the MTs are themselves a matter of considerable choice in starting point, and often beyond the mathematical education of the teacher. What is real is hard to see, while standards flash brightly from every test, text and exhortation that comes the teacher's way.

Learning trajectory research develops evidence and evidence based trajectories (LTs). Evidence establishes that LTs are real for some students, a possibility for any student and possibly modal trajectories for the distribution of students. LTs are too complex and too conditional to serve directly as standards. Still, LTs point the way to optimal learning sequences and warn against hazards that could lead to sequence errors (see below). The CCSS made substantial use of LTs, but standards cannot simply be LTs; standards have to include the essential mathematics, MTs, whether we know anything about its location in an LT or not, and standards have to accommodate the variation in students, if not teachers, at each grade level.

How do and could these four trajectories (LTs, MTs PLoTs, and SSTs) interact? A system could just leave it to individual teachers to reckon the optimization among them. It could impose strong SSTs as pressure in an accountability system, without providing for PLoTs or taking advantage of LTs. It could name the territory between what students bring (PLoTs) and the what standards demand (SST) the "achievement gap", a dark void that only explains steps not taken rather than which way to go. It could tell teachers to keep turning the pages of the textbook based on standards according to the planned pace, and rely on the shear force of expectation to pull students along. At least this would create the opportunity to learn, however fleeting and poorly prepared students might be to take advantage of it. While this is better than denial of opportunity, it is a hollow, if not cynical, response to the promise standards make to students. Shouldn't we do better?

What would be better? Some nations, including high performing nations, assume in the structure of their instructional systems that students differ at the beginning of each lesson. Asian classrooms, K-5, and mostly 6-9, follow a daily trajectory of initially projecting the divergence of students' development (refracted through the day's mathematics problem/s) into the classroom discourse and pulling the divergence toward a convergent learning target. The premise is: each lesson begins with divergence and ends with convergence. Such a system requires enough time to achieve convergence each day, enough time on a small number of problems. A hurried instructional system cannot 'wait' for students each day. Standards must require less to learn rather than more each year to make time for daily convergence. A system which optimises daily convergence will be more robust and accumulate less debt in the form of students unprepared for the next lesson. Such debt compounds. Unlike the national debt, it does not compound quietly, but makes all the noises of childhood and adolescence scorned.

Start by understanding the task and then the people in place who can do their parts to accomplish the task. The task is to take the domain of PLoTs, the given rough terrain of what the distribution of students bring, and transform the PLoTs to SSTs, give or take. The function that can take PLoTs to SSTs is mapped by the LTs and MTs. That is, LTs and MTs can provide the map from PLoTs to SSTs . The map, alas, is of a territory that is only partially explored. There are still unknown seas and fears of sea monsters and dreams of gold to frighten and distract us from the voyage. Still, we know enough in elementary grades to do what is needed to make LTs a part of teacher knowledge and a feature in tools for teachers.

Teachers need knowledge of how LTs work and the specifics of LTs that will help them understand the most common PLoTs they will find among their students (Murata, A., & Fuson, K. C., 2006). They need knowledge of the relevant MTs. And they need tools that illuminate rather than obscure the PLoTs. They need instructional programs and lesson protocols that pose SSTs as the finish line, but accommodate PLoT variation. They need time within the lesson and across the unit to pull students from PLoTs along LTs to the SSTs. This requires standards to be within reach.

The crucial issue in this situation is how well the standards driven texts and tests improve the performance of the instructional system in moving the PLoTs along the LTs. It is quite possible for standards to be out of whack with LTs and PLoTs so that they diminish performance. Standards are only a good idea when they usefully map underlying LTs and MTs so they can help teachers see and respond to PLoTs. If the sequence in the standards conflicts seriously with LTs or are too far removed from PLoTs, they can steer the instructional systems away from teaching and learning, toward statuesque poses facing out and the same waste of chances inside.

For example, the CCSS at grade 7 have a standard for proportional relationships.

2. Recognize and represent proportional relationships between covarying quantities.

- **a.** Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.
- **b.** Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
- **c.** Represent proportional relationships by equations. *For example, total cost, t, is proportional to the number, n, purchased at a constant price, p; this relationship can be expressed as* $t = pn$.
- **d.** Explain what a point (x, y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0, 0) and $(1, r)$ where r is the unit rate.

This standard is the culmination of a manifold of progressions and, itself, the beginning of more advanced progressions. Pat Thompson has remarked (2010, advice to standards) that proportionality cannot be a single progression because it is a whole city of progressions. This standard, which stands along side other standards on ratios and rates, explicitly draws on prior knowledge of fractions, equivalence, quantitative relationships, coordinate graph, unit rate, tables, ratios, rates and equations. Implicitly, this prior knowledge grows from even broader prior knowledge. The sequence supporting this Standard in the SST barely captures the peaks of a simplification of the knowledge structure. The complexity of the manifold of LTs guarantees that the distribution of PLoTs in a classroom will have splendid variety.

What could help the teacher confronted with the variety of readiness? Certainly not pressure to "cover" the standards in sequence (SST), keep moving along at a good pace to make sure all students have an 'opportunity' to see every standard flying by. Perhaps some knowledge of the LTs would help teachers understand the variety of PLoTs and what direction to lead the students from wherever they begin the lesson. Even hypothetical LTs can do more good than harm because they conceptualize the student as a competent knower and learner in the process of learning and knowing more (Clements, 2004a). Perhaps a system of problems and assignments with the diagnostic value of revealing how different students see the mathematics…how they think about it…where they are along the LT. A teacher needs the thinking itself, not a score that evaluates the thinking.

How do standards express the form and substance of what students learn?

What is the nature of the 'things' students learn? Sometimes what is wanted is a performance, as in learn to ride a bike. Standards, instruction and assessment can happily focus on the visible performance in such cases. But often, in mathematics anyway, is a mental action on a mental object, reasoning maneuvers and rules, representational systems and languages for mathematical objects and relations, cognitive schema and strategies, webs of structured knowledge, and social representations, and so on. Many of these learned things are systems that interact with other systems in thinking, knowing and doing. Standards cannot express this kind of complexity; they refer to some observable surface of learning. But this linguistic convenience can lead to logical fallacies when we attribute unwarranted 'thinginess' properties to what we actually want students to learn.

The important point is that learned things are not things or topics (names) and not just standards. A sequence of topics or standards skims the surface and misses the substance and even the form of a subject. Compare, for example, the Standard,

- **Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators.** *For example,* $^2/_3 + {}^5/_4 = {}^8/_{12} + {}^{15}/_{12} = {}^{23}/_{12}$. *(In general,* $^a/_b + {}^c/_d = {}^{(ad + bc)}/_{bd}$*.)*

to what the student must actually know and do to "meet" the standard (for example, Steffe, 2004,2009; Confrey et al, 2008, 2009; Wu, 2007; Saxe et al, 2005). The standard gives a goal, but does not characterize the knowledge and competencies needed to achieve the goal. While this point may seem obvious, it gets lost in the compression chambers where systems are organized to manage instruction for school districts. Devices are installed to manage "pacing" and monitor progress with "benchmark assessments". These devices treat the grade level standards as the form and substance of instruction. That is, students are taught grade level "standards" instead of mathematics. This nonsense is actually widespread, especially where pressures to "meet standards" are greatest.

Standards use conventional names and phrases for topics in a subject. To what do these refer?

If the field had a well understood corpus of cognitive actions, situations, knowledge etc. then these names could refer to parts of this corpus. But the field, school mathematics, has no such widely understood corpus (indeed, it is an important hope that common standards will lead to common understandings like this). What the names refer to, in effect, are the familiar conventions of what goes on in the classrooms. The reference degenerates to the old habits of teaching: assignments, grading, assessment, explanation, discussion. The standards say, 'Do the usual assortment of classroom activities for some content that can be sorted into the names in the standards. We will call this "covering the standards" with instructional activity.

"Covering" has a very tenuous relationship with learning. First, there are many choices within a topic about focus, coherence within and between topics, what students should learn to do with knowledge, how skillful they need to be at what, and so on endlessly. Teachers make these choices in many different ways. Too often, the choices are made in support of a classroom behavior management scheme relied on by the teacher. Second, different students will get very different learning from the same offered activity. Third, the quality of the discussion, the assigned and produced work, the feedback given to students will vary widely by teacher working under the blessing of the same standard.

Covering is at best weak. When combined with standards that are too far from the prior knowledge of students, and too many; the chemistry gets nasty in a hurry. Teachers move on without the students; students accumulate debts of knowledge (knowledge owed to them) and opportunities for understanding the next chapter, the next course are undermined.

The foregoing discussion of instructional systems illustrates the importance (and potential for mayhem) in sequencing standards. What constituents are necessary and sufficient as prior knowledge for a given concept or action, and how can the constituents be arranged to lead up to the target concept? This question has many local answers that have to be fitted together into regions that make some sense, if not harmony. Standards are further constrained by how much can be learned at any one grade level, and by the coherence within a grade level. These questions are not only design choices, but potential sources of error with consequences for the viability of instruction. The next sections examine the types of errors that could menace a standards based system.

Types of Sequence Errors

There are several types of errors with serious consequences for students and teachers in the way standards might be sequenced. For example, a common type of sequence error occurs when a concept, B depends on A2 version of concept A, more evolved than the A1 version; Standards have only developed A1. Student tries to learn B using A1 instead of A2. Rate, proportional relationships and linearity (B) depend on understanding multiplication as a scaling comparison (version A2), but students may have only developed version A1 concept of multiplication, the total of things in a groups of *b* each.

In the CCSS, multiplication is defined in grade 3 as $a \times b = c$ means a groups of *b* things each is *c* things. In grade 4, the concept of multiplication is extended to comparison where $c = a \times b$ means *c* is a times larger than *b*. In grade 5, the CCSS has:

5. Interpret multiplication as scaling (resizing), by:
   a. Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.
   b. Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n\times a)/(n\times b)$ to the effect of multiplying a/b by 1.

In grade 6 and 7 rate, proportional relationships and linearity build upon this scalar extension of multiplication. Students who engage these concepts with the unextended version of multiplication (*a* groups of *b* things) will have PLoTs that do not support the required MTs. This burdens the teacher and student with recovering through LTs. This will be taxing enough without ill sequenced standards causing instructional systems to neglect extending multiplication.

Major types of sequence errors follow:

1. Unrealistic:
   a. Too much too fast so gaps in learning create sequence issues for students, system cannot deliver students who are in sequence.
   b. Distribution of prior mathematics knowledge and proficiency in the student and teacher population is too far from the standards; no practical way to get students in a good enough sequence.
2. Missing ingredient:
   a. A is an essential ingredient of B, Standards sequence B before A.
   b. Coherence requires progression ABC, but standards only have AC
   c. Term is used that has insufficient definition for that use.
3. Cognitive prematurity:
   a. B depends on cognitive actions and structures that have not developed yet.
   b. B is a type of schema or reasoning system, learner has not developed that type of schema or system.
   c. Student develops immature version of B and carries it forward (see 4)
4. Contradiction:
   a. Cognitive development entails ABC, mathematical logic entails CBA.
5. Missing connection: B is about or depends on connection between X-Y , but X-Y connection not established.
6. Interference:
   a. B depends on A2 version of A, more evolved than A1 version; Standards have only developed A1. Student tries to learn B using A1 instead of A2.
   b. B belongs nestled between A and C, but D is already nestled there. When learning B is attempted, D interferes.
7. Cameo:
   a. B is learned but not used for a long time. There is no C such that C depends on B for a long time. B makes a cameo appearance and then gets lost in the land of free fragments.
8. Hard Way:
   a. C needs some ideas from B, but not all the difficult ideas and technical details that make B

take more time than it is worth and make it hard for students to find the needed ideas from B, so C fails.

b. There are multiple possible routes to get from A to E, standards take an unnecessarily difficult route

9. Aimless:

a. Standards presented as lists that lack comprehensible progression.

Types of Focus and Coherence Errors

The issues of focus and coherence in standards deserves more attention than we will give it here. Nonetheless, learning trajectories interact with coherence and focus in standards. The following are critical types of error of focus and coherence:

1. Sprawl:

   a. Mile wide, inch deep. Collection of standards dilutes the importance of each one.

   b. Standards demand more than is possible in the available time for many students and teachers, so teachers and students forced to edit on the fly. This is the opposite of focus.

   c. Standards are just lists without enough organisational cues in relation to hierarchy of concepts and skills

2. Wrong grain size

   a. The granularity is too specific or too general. The important understanding is at a certain level of specificity where the structure and the cognitive handles are, more specific or more general; grain size will not match up to prior knowledge, mental objects and actions on them (see Aristotle *Ethics*: the choice of specificity is a claim that should be explicit and defended.)

   b. Too fine: complex ideas are chopped up so the main idea is lost; the coherence may be evoked, but not illuminated. Alignment transactions in test construction, materials development miss the main point but 'cover' the incidentals. Students can perform the vertical line test but do not know what a function is or how functions model phenomena.

   c. Too broad: includes whatever and focuses on nothing in particular.

3. Wrong focus

   a. Focus on answer getting methods, often mnemonic devices, rather than mathematics.

4. Narrow focus

   a. Just skills, or just concepts or just process; or just two out of three.

5. Priorities do not cohere:

   a. Fragments that have large gaps between them;

   b. grain size too fine

6. Congestion:

   a. Some grade levels are congested with too much to be learned; density precludes focus

   b. B, C, D are all being learned at once, but cognitive actions needed for learning can only handle one or two at a time. Only BC and CD are learned, but the essential point is learning BCD and the system BC-BD-CD.

7. Inelegance:

   a. AXBYCZ is equivalent to ABC and wasted time and cognition on –X-Y-Z.

8. Waste:

   a. Invest time and cognition on B and B is not important.

9. Resolution of hierarchy:

   a. The hierarchal relationship between standards is not explicated. Details are confused with main ideas.

   b. The hierarchy of standards does not explain relationships among ideas, it just collects standards into categories.

10. Excessively literal reading:

   a. This error is in the reading as much as the writing; it leads to fragmented interpretation of the subject, losing the coherence between the standards.

   b. Reading individual standards as individual ingredients of a test. when the explicit goal is to have the ingredients cook into a cake, tasting the uncooked ingredients is a poor measure of how the cake tastes (although it is related). The goal, as stated in the grade level introductions and the practices standards is for the students to cook.

What are Standards?

Standards are promises. Standards promise the student, "Study and learn what is here, do your assignments and we promise you will do well on the test." We need tests and examinations designed to keep that promise. We need school systems designed to keep the promises.

Bibliography

Baroody, A. J., Cibulskis, M., Lai, M.-l., & Li, X. (2004). Comments on the use of learning trajectories in curriculum development and research. *Mathematical Thinking and Learning. 6*(2), 227–260.

Black, P., & Wiliam, D. (1998). Assessment and classroom learning. *Assessment in Education, 5*(1),

Black, P., & Wiliam, D. (1998). "Inside the Black Box: Raising Standards Through Classroom Assessment," *Phi Delta Kappan 80*.

Case, R., & Okamoto, Y. (1996). The role of central conceptual structures in the development of children's thought. *Monographs of the Society for Research in Child Development*. Serial No. 246, 61, (1-2),

Clements, D.H. (1999c). Teaching length measurement: Research challenges. *School Science and Mathematics, 99*(1), 5-11.

Clements, D. H., & Sarama, J. (2004a). Hypothetical learning trajectories. *Mathematical Thinking and Learning, 6*(2).

Clements, D. H., & Sarama, J. (2004b). Learning trajectories in mathematics education. *Mathematical Thinking and Learning. 6*(2), 81-89.

Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. New York: Routledge.

Common Core State Standards for Mathematics, 2010; Corestandards.org

Confrey, J., Maloney, A., Nguyen, K., Mojica, G., & Myers, M. (2009). *Equipartitioning/Splitting as a Foundation of Rational Number Reasoning Using Learning Trajecories*. Paper presented at the 33rd Conference of the International Group for the Psychology of Mathematics Education, Thessaloniki, Greece.

Confrey, J. (2008). *A synthesis of the research on rational number reasoning: A learning progressions approach to synthesis*. Paper presented at the 11th International Congress of Mathematics Instruction.

Fuson, K.C. (2004). Pre-K to grade 2 goals and standards: Achieving 21st-century mastery for all. In D.H. Clements, J. Sarama, and A. DiBiase (Eds.), *Engaging Young Children in Mathematics* Mahwah, NJ: Erlbaum.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Murata, A., & Fuson, K. C. (2006). Teaching as assisting individual constructive paths within an interdependent class learning zone: Japanese first graders learning to add using 10. *Journal for Research in Mathematics Education, 37*, 421-456.

NCTM. (2006). *Curriculum focal points for prekindergarten through grade 8 mathematics: A quest for coherence*. Reston, VA: National Council of Teachers of Mathematics.

Nührenbörger, M. (2001). Insights into children's ruler concepts—Grade-2 students' conceptions and knowledge of length measurement and paths of development. In M.V.D. Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology in Mathematics Education, 3*, 447-454. Utrecht, The Netherlands: Freudenthal Institute.

Nunes, T., Light, P., and Mason, J.H. (1993). Tools for thought: The measurement of length and area. *Learning and Instruction, 3*, 39-54.

Park, J., & Nunes, T. (2001). The development of the concept of multiplication. *Cognitive Development, 16*, 763-773.

Peterson, P. L., Carpenter, T. P., & Fennema, E. H. (1989). Teachers' knowledge of students' knowledge in mathematics problem solving: Correlational and case analyses. *Journal of Educational Psychology, 81*, 558-569.

Sarama, J., & Clements, D. H. (2002). *Building Blocks* for young children's mathematical development. *Journal of Educational Computing Research, 27*(1&2), 93-110.

Sarama, J., & Clements, D. H. (2009a). *Early childhood mathematics education research: Learning trajectories for young children*. New York: Routledge.

Saxe, G., Taylor, E., McIntosh, C., & Gerhart, M. (2005). Representing Fractions with standard notions: A developmental analysis. *Journal for research in Mathematics Education. 36* (2), 137-157.

Sherin, B., & Fuson, K. (2005). Multiplication Strategies and the Appropriation of Computational Resources. *Journal for Research in Mathematics Education., 36* (4), 347-395.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Steffe, L. P. (2004)."On the construction of learning trajectories of children: The case of commensurate fractions". *Mathematical Thinking and Learning. 6* (2), 129-162

Steffe, L. P. & Olive, J (2009). *Children's fractional knowledge*. Boston, Springer.

Wilson, P. H. (2009). *Teachers' Uses of a Learning Trajectory for Equipartitioning*. North Carolina State University, Raleigh, NC.

Wu, H. (2007), *Fractions, decimals and rational numbers", http://math.berkeley.edu/-wu/*